Proportional Shapley levels values

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Abstract

The proportional Shapley value (Besner 2016; Béal et al. 2017; Gangolly 1981) is an proportional counterpart to the Shapley value (Shapley 1953b) in cooperative games. As shown in Besner (2017a), the proportional Shapley value is a convincing non-linear alternative, especially in cost allocation, if the stand alone worths of the players are plausible weights.

To enable similar properties for cooperative games with a level structure, we generalize this value. Therefore we adapt the proceeding applied to the weighted Shapley values in Besner (2017b). We present, analogous to the four classes of weighted Shapley levels values in Besner (2017b), four different values, the proportional Shapley hierarchy levels value, the proportional Shapley support levels value, the proportional Shapley alliance levels value and the proportional Shapley collaboration levels value, respectively.

Keywords Cooperative game · Level structure · (Proportional) Shapley (levels) value · Proportionality · Component substitution · Dividends

1 Introduction

Winter (1989) introduced a model, called level structures, for hierarchical cooperation structures, like in political organisations, governmental authorities or hierarchical organized groups, and presented the Shapley levels value for cooperative games with a level structure. This value generalizes the Shapley value (Shapley 1953b) and the Owen value (Owen 1977), respectively. The Shapley levels value distributes in the sum the same payoff to the players of symmetric components which are subsets of the same component in an induced game where some components are the players.

Often coalitions have not the same power if they act as players. If there exist related exogenous given weights, the weighted Shapley levels values, presented in Besner (2017a) can used for games with a level structure.

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But sometimes the weights are inherent to the coalition function, for example in cost games. Besides the proportional rule (Moriarity 1975), the proportional value (Feldman 1999; Ortmann 2000) provides one possibility to divide benefits or costs in cooperative games in a proportional manner, regarding the worths of the players. Huettner (2015) generalized this value to games with a coalition structure[1]. Recently, there was a rediscovery of the proportional Shapley value (Besner 2016; Béal et al. 2017; Gangolly 1981), a weighted value where the worths of the singletons are the weights. Thus, this value takes proportionality into account too. Convincing axioms and characterizations (Besner 2017a; Béal et al. 2017) recommend this value for games where the weights are the stand alone worths of the players.

It speaks for itself, therefore, to generalize the proportional Shapley value to cooperative games with a level structure. We proceed thereby in analogy to the weighted Shapley levels values, presented in Besner (2017b)[2]: if in Besner (2017b, algorithm 5.1) the weights for the coalitions, acting as players in the used weighted Shapley values, are replaced by the worths of the coalitions, we obtain the proportional Shapley hierarchy levels value and, if we replace in the definitions of the other three classes the weights of the coalitions by the worths of the coalitions, we receive the proportional Shapley support levels value, the proportional Shapley alliance levels value and the proportional Shapley collaboration levels value, respectively.

The proportional Shapley hierarchy levels value can be characterized by efficiency and proportional balanced group contributions, a proportional variant of balanced group contributions (Calvo, Lasaga and Winter 1996) which can be used, together with efficiency, to characterize the Shapley levels value. This first value doesn't satisfy dummy. All four values are efficient and non-linear. Therefore, in contrast to the respective weighted Shapley levels values, the proportional Shapley support levels value, the proportional Shapley alliance levels value and the proportional Shapley collaboration levels value are not in the Harsanyi set (Hammer 1977; Vasil'ev 1978), but meet also dummy. They satisfy only weaker forms of additivity and thus the characterizations of the related weighted Shapley levels values are not transferable one to one. Nevertheless, with a new extension of the dummy axiom, called loyalty, we get comparable axiomatizations. In the case of the proportional Shapley support levels value we present also a characterization which uses a component substitution property, extending the player splitting axiom in Besner (2017a). The opening words from each related chapter in Besner (2017b) can be adapted, so we keep it short.

The plan of this paper is derived from Besner (2017b) and is organized as follows. Section 2 contains the preliminaries and section 3 presents new properties and used axioms. In the following sections we introduce the values, in section 4 the proportional Shapley hierarchy levels value, in section 5 the proportional Shapley support levels value, in section 6 the proportional Shapley alliance levels value and in section 7 the Shapley collaboration levels value. Section 8 gives a conclusion. An appendix (section 9) provides all the proofs.

[1]In Besner (2016, subsection 5.5.1) this value is extended to games with a level structure.

[2]All new values in this paper are special cases of classes of values for level structures, proposed first in Besner (2016), which also contain the weighted Shapley levels values given in Besner (2017b).

2 Preliminaries

We denote by $\mathbb{R}$ the real numbers, by $\mathbb{R}_{++}$ the set of all positive real numbers and by $\mathbb{Q}_{++}$ the set of all positive rational numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players, and denote by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. We refer to a TU-game also only by v. The subsets $S \subseteq N$ are called **coalitions**, $v(S)$ is the **worth** of coalition S and the set of all nonempty subsets of S is denoted by Ω^S. The set of all TU-games with player set N is denoted by $\mathcal{G}^N$ and, if $v(\{i\}) > 0$ for all $i \in N$, by $\mathcal{G}_0^N$. The **restriction** of (N, v) to the player set $S \in \Omega^N$ is denoted by (S, v).

Let $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and $S \subseteq N$. The **dividends** $\Delta_v(S)$ (Harsanyi 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } S \in \Omega^N, \text{ and} \\ 0, & \text{if } S = \emptyset. \end{cases} \tag{1}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. We call a coalition $S \subseteq N$ **active** in v if $\Delta_v(S) \neq 0$. Player $i \in N$ is called a **dummy player** if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $S \subseteq N \backslash \{i\}$; players $i, j \in N$, $i \neq j$, are called **symmetric** in v, if $v(T \cup \{i\}) = v(T \cup \{j\})$, and **weakly dependent** (Besner 2017a) if $v(T \cup \{k\}) = v(T) + v(\{k\})$, $k \in \{i, j\}$, for all $T \subseteq N \backslash \{i, j\}$.

A partition $\mathcal{B}$ of the player set N is called a **coalition structure** on N. Each $B \in \mathcal{B}$ is called a **component** and $\mathcal{B}(i)$ denotes the component that contains a player $i \in N$. A **level structure** (Winter 1989) on N is a finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N such that:

- $\mathcal{B}^0 = \big\{\{i\}\colon i \in N\big\}$.
- $\mathcal{B}^{h+1} = \{N\}$.
- For each r, $0 \leq r \leq h$, $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$, i. e. $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$.

$\mathcal{B}^r$ is called the r-th **level** of $\underline{\mathcal{B}}$; $\overline{\mathcal{B}}$ is the set of all components $B \in \mathcal{B}^r$ of all levels $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$; $\mathcal{B}^r(i)$ is called the **r-th component**, $\mathcal{B}^h(i)$ the **top component**[3] containing player $i \in N$ and, if $S \in \Omega^N$ is a subset of a component of the r-th level, $\mathcal{B}^r(S)$ is the component of the r-th level which contains the coalition S.

The collection of all level structures with player set N is denoted by $\mathcal{L}^N$. A TU-game $(N, v) \in \mathcal{G}^N$ together with a level structure $\underline{\mathcal{B}} \in \mathcal{L}^N$ is an **LS-game** $(N, v, \underline{\mathcal{B}})$. If N and $\underline{\mathcal{B}}$ are clear we refer to an LS-game also only by v. The set of all LS-games on N is denoted by $\mathcal{GL}^N$, by $\mathcal{GL}_0^N$ if $v(B) > 0$ for all $B \in \overline{\mathcal{B}}$, by $\mathcal{GL}_{0_\mathbb{Q}}^N$ if $v(B) \in \mathbb{Q}_{++}$ for all $B \in \overline{\mathcal{B}}$, and by $\mathcal{GL}_{0+}^N$ if $v(S) > 0$ for all $S \in \Omega^B$ and all $B \in \mathcal{B}^h$.

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$ and $T \in \Omega^N$:

- From a level structure on N follows a **restricted** level structure on T by eliminating the players in $N \backslash T$. With coalition structures $\mathcal{B}^r|_T := \{B \cap T\colon B \in \mathcal{B}^r,\ B \cap T \neq \emptyset\}$,

[3] In some respects we consider the $(h+1)$-th level not as an regular level of the level structure.

$0 \le r \le h{+}1$, the new level structure on T is given by $\underline{\mathcal{B}}|_T := \{\mathcal{B}^0|_T, ..., \mathcal{B}^{h+1}|_T\} \in \mathcal{L}^T$ and $(T, v, \underline{\mathcal{B}}|_T) \in \mathcal{GL}^T$ is called the **restriction** of $(N, v, \underline{\mathcal{B}})$ to player set T.

- We denote by $\mathcal{B}^r_{\mathcal{I}}|_T$, $0 \le r \le h+1$, the coalition structure on T, given by

$$\mathcal{B}^r_{\mathcal{I}}|_T := \begin{cases} \{T\}, \text{ if } r = h+1, \\ \big\{B \in \overline{\mathcal{B}} : \ B \subseteq (B^r \cap T),\ B^r \in \mathcal{B}^r,\ B \not\subseteq B' \in \mathcal{B}, B' \subseteq (B^r \cap T)\big\}, \text{ else.} \end{cases}$$

  With the level structure $\underline{\mathcal{B}_{\mathcal{I}}}|_T = \{\mathcal{B}^0_{\mathcal{I}}|_T, ..., \mathcal{B}^{h+1}_{\mathcal{I}}|_T\} \in \mathcal{L}^T$ the LS-game $(T, v, \underline{\mathcal{B}_{\mathcal{I}}}|_T) \in \mathcal{GL}^T$ is called the **internally induced restriction** of $(N, v, \underline{\mathcal{B}})$ to player set T.

- We define $\underline{\mathcal{B}^r} := \big\{\mathcal{B}^{r^0}, ..., \mathcal{B}^{r^{h+1-r}}\big\} \in \mathcal{L}^{\mathcal{B}^r}$, $0 \le r \le h$, as the **induced r-th level structure** from $\underline{\mathcal{B}}$ by considering the components $B \in \mathcal{B}^r$ as players, where $\mathcal{B}^{r^k} := \big\{\{B \in \mathcal{B}^r : B \subseteq B'\} : B' \in \mathcal{B}^{r+k}\big\}$, $0 \le k \le h+1-r$. If $T = \bigcup_{B \subseteq T} B$, $B \in \mathcal{B}^r$, and we want to stress this property, T is denoted by T^r. Each such T^r is related to a coalition of all players $B \in \mathcal{B}^r$, $B \subseteq T^r$, in the induced r-th level structure, denoted by $\mathcal{T}^r := \{B \in \mathcal{B}^r : B \subseteq T^r\}$ and vice versa. The induced **r-th level game** $\big(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}\big) \in \mathcal{GL}^{\mathcal{B}^r}$ is given by

$$v^r(\mathcal{T}^r) := v(T^r) \text{ for all } \mathcal{T}^r \in \Omega^{\mathcal{B}^r}. \tag{2}$$

- We define $\underline{\mathcal{B}_r} := \big\{\mathcal{B}^0, ..., \mathcal{B}^r, \{N\}\big\} \in \mathcal{L}^N, 0 \le r \le h$, as the **$r$-th cut level structure** from $\underline{\mathcal{B}}$ if we cut out all levels between the r-th and the $(h+1)$-th level. $(N, v, \underline{\mathcal{B}_r})$ is called the **r-th cut** of $(N, v, \underline{\mathcal{B}})$.

Note that we have for each $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ also $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, each TU-game (N, v) corresponds to an LS-game $(N, v, \underline{\mathcal{B}_0})$ with a **trivial level structure** $\underline{\mathcal{B}_0} := \{\mathcal{B}^0, \mathcal{B}^1\}$ and $\overline{\mathcal{B}_r}$ denotes the set of all components $B \in \mathcal{B}^\ell$ of all levels $\mathcal{B}^\ell \in \underline{\mathcal{B}}$, $0 \le \ell \le r$.

A **TU-value** ϕ is an operator that assigns to any $v \in \mathcal{G}^N$, $N \in \mathcal{N}$, a payoff vector $\phi(N, v) \in \mathbb{R}^N$, an **LS-value** φ is an operator that assigns payoff vectors $\varphi(N, v, \underline{\mathcal{B}}) \in \mathbb{R}^N$ to all LS-games $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$. The **Shapley value** Sh (Shapley 1953b) is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all } \ i \in N.$$

Let $\mathrm{W} := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$ with $\mathrm{w}_i := \mathrm{w}(i)$ for all $\mathrm{w} \in \mathrm{W}$ and $i \in \mathfrak{U}$. For $\mathrm{w} \in \mathrm{W}$, the (simply) **weighted Shapley value**[4] Sh^{w} (Shapley 1953a) is given by

$$Sh^{\mathrm{w}}_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\mathrm{w}_i}{\sum_{j \in S} \mathrm{w}_j} \Delta_v(S) \quad \text{for all } \ i \in N.$$

Let $N \in \mathcal{N}$ and $v \in \mathcal{G}^N_0$. The **proportional Shapley value** Sh^p (Besner 2016; Béal et al. 2017; Gangolly 1981) is given by

$$Sh^p_i(v) := \sum_{S \subseteq N,\, S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) \quad \text{for all } \ i \in N.$$

[4]We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987)

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$ and for all $T \subseteq N$, $T \ni i$,

$$K_T(i) := \prod_{r=0}^{h} K_T^r(i), \quad \text{where}$$
$$K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \cap T \neq \emptyset\}|}.$$

The **Shapley levels value** Sh^L (Winter 1989) is given by[5]

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_T(i)\Delta_v(T) \text{ for all } i \in N.$$

3 Axioms and Properties

We introduce two new properties for players. The first one plays a basic role in this paper and is an extension of a dummy player.

Definition 3.1. *For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$ and $v \in \mathcal{GL}^N$ a player $i \in N$ is called a **loyal player** (to the top component) in $(N, v, \underline{\mathcal{B}})$, if*

$$MC_i^v(S \cup T) = MC_i^v(S) \text{ for all } S \subseteq \mathcal{B}^h(i)\backslash\{i\},\, T \subseteq N\backslash\mathcal{B}^h(i). \tag{3}$$

It is evident, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, that a loyal player coincides with a dummy player who is only loyal to herself. A loyal player is loyal to the top component containing her in the sense that she is not interested to join a coalition outside the top component. In all such coalitions she acts only passively. The following lemma stresses the naming of a loyal player: outside the top component containing the loyal player all coalitions where the loyal player is a member are not active in v.

Lemma 3.2. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$. Player $i \in N$ is a loyal player in $(N, v, \underline{\mathcal{B}})$, iff $\Delta_v(R) = 0$ for all $R \subseteq N$, $R \not\subseteq \mathcal{B}^h(i)$, $R \ni i$.*

For the proof, see appendix 9.1. The second new property generalizes weakly dependent.

Definition 3.3. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $0 \leq r \leq h$, and $v \in \mathcal{GL}^N$. Players $i, j \in N$, $i \neq j$, are called **dependent outside the r-th component** (short **rc-dependent**) in $(N, v, \underline{\mathcal{B}})$, if*

$$MC_k^v(S_k) = MC_k^v\big(\mathcal{B}^r(k) \cap S_k\big) \text{ for all } S_k \in \Omega^{N\backslash\{i,j\}}, S_k \not\subseteq \mathcal{B}^r(k), k \in \{i, j\}.$$

*If $r = h$ we call the players **dependent outside the top component** (short **tc-dependent**).*

Obviously dependent outside the zeroth component coincides with weakly dependent. This definition has the interpretation that an rc-dependent player is only interested to join a coalition outside the r-th component if all rc-dependent players are in the joined coalition. The following lemma reveals this relationship.

[5] This formula is presented in Calvo, Lasaga and Winter (1996, eq. (1)).

Lemma 3.4. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $0 \leq r \leq h$, and $v \in \mathcal{GL}^N$. Players $i, j \in N$, $i \neq j$, are rc-dependent in $(N, v, \underline{\mathcal{B}})$, iff $\Delta_v(S_k \cup \{k\}) = 0$ for all $S_k \in \Omega^{N\backslash\{i,j\}}$, $S_k \not\subseteq \mathcal{B}^r(k), k \in \{i, j\}$.*

For the proof, see appendix 9.2. Now we can present the axioms[6].

Efficiency, E. For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$, we have $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$.

Dummy, D. For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ and $i \in N$ a dummy player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = v(\{i\})$.

Loyalty, L[7]. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$ and $i \in N$ a loyal player in v, we have

$$\varphi_i(N, v, \underline{\mathcal{B}}) = \begin{cases} v(\{i\}), \text{ if } h = 0, \\ \varphi_i\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big), \text{ else.} \end{cases}$$

Dummy player out, DO[8] For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $j \in N$ a dummy player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i\big(N\backslash\{j\}, v, \underline{\mathcal{B}}|_{N\backslash\{j\}}\big)$ for all $i \in N\backslash\{j\}$.

Internal (induced restriction) **dummy player out, IDO[8].** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $j \in N$ a dummy player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i\big(N\backslash\{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N\backslash\{j\}}\big)$ for all $i \in N\backslash\{j\}$.

Loyal player out, LO[8]. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $j \in N$ a loyal player in v and $i \in N\backslash\{j\}$, we have

$$\varphi_i(N, v, \underline{\mathcal{B}}) = \begin{cases} \varphi_i\big(N\backslash\{j\}, v, \underline{\mathcal{B}}|_{N\backslash\{j\}}\big), \text{ if } \mathcal{B}^h(i) \neq \mathcal{B}^h(j), \\ \varphi_i\big(N\backslash\{j\}, v, \underline{\mathcal{B}}|_{N\backslash\{j\}}\big) + \varphi_i\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big) \\ \qquad -\varphi_i\big(\mathcal{B}^h(i)\backslash\{j\}, v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)\backslash\{j\}}\big), \text{ else.} \end{cases}$$

Internal (induced restriction) **loyal player out, ILO[8].** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $j \in N$ a loyal player in v and $i \in N\backslash\{j\}$, we have

$$\varphi_i(N, v, \underline{\mathcal{B}}) = \begin{cases} \varphi_i\big(N\backslash\{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N\backslash\{j\}}\big), \text{ if } \mathcal{B}^h(i) \neq \mathcal{B}^h(j), \\ \varphi_i\big(N\backslash\{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N\backslash\{j\}}\big) + \varphi_i\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1\mathcal{I}}}|_{\mathcal{B}^h(i)}\big) \\ \qquad -\varphi_i\big(\mathcal{B}^h(i)\backslash\{j\}, v, \underline{\mathcal{B}_{h-1\mathcal{I}}}|_{\mathcal{B}^h(i)\backslash\{j\}}\big), \text{ else.} \end{cases}$$

Homogeneity, H. For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ and scalars $\alpha \in \mathbb{R}$, we have $\varphi(N, \alpha v, \underline{\mathcal{B}}) = \alpha\varphi(N, v, \underline{\mathcal{B}})$.

Weak LS-additivity, WA[9]. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$, $v'(B) = c \cdot v(B)$ for all $B \in \overline{\mathcal{B}}$, $c > 0$, we have

$$\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}}).$$

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[6] In the case of using a subdomain, we require an axiom to hold when all games belong to this subdomain.
[7] This is also an extension of dummy.
[8] These axioms extend dummy player out (Tijs and Driessen, 1986) which is related to null player out in Derks and Haller (1999).
[9] These are extensions of weak additivity (Besner 2017a).

Completely weak LS-additivity, CWA[9]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$, $v'(S) = c \cdot v(S)$ for all $S \subseteq B$, $B \in \mathcal{B}^h$, $c > 0$, we have

$$\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}}).$$

Proportional balanced group contributions, PBGC[10]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$ and $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, we have

$$\frac{\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum_{i \in B_k} \varphi_i(N \backslash B_\ell, v, \underline{\mathcal{B}}|_{N \backslash B_\ell})}{v(B_k)} = \frac{\sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum_{i \in B_\ell} \varphi_i(N \backslash B_k, v, \underline{\mathcal{B}}|_{N \backslash B_k})}{v(B_\ell)}.$$

Symmetry between components, SymBC[11] **(Winter 1989).** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, and B_k, B_ℓ are symmetric in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}^{\mathcal{B}^r}$, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

Proportionality between components, PBC[12]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, and B_k, B_ℓ are weakly dependent in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}_0^{\mathcal{B}^r}$, we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{v(B_k)} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{v(B_\ell)}.$$

Proportionality within components, PWC[12]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are *rc*-dependent in v, we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{v(B_k)} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{v(B_\ell)}.$$

Proportional top components contributions, PTCC[12]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$, $k, \ell \in N$, $\mathcal{B}^r(k), \mathcal{B}^r(\ell) \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and all $i \in \mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)$ are *tc*-dependent in v, we have

$$\frac{\varphi_k(N, v, \underline{\mathcal{B}})}{v(\{k\})} = \frac{\varphi_\ell(N, v, \underline{\mathcal{B}})}{v(\{\ell\})}, \text{ if } h = 0, \text{ and}$$

$$\sum_{i \in \mathcal{B}^r(k)} \frac{\varphi_i(N, v, \underline{\mathcal{B}}) - \varphi_i\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big)}{v\big(\mathcal{B}^r(k)\big)} = \sum_{i \in \mathcal{B}^r(\ell)} \frac{\varphi_i(N, v, \underline{\mathcal{B}}) - \varphi_i\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big)}{v\big(\mathcal{B}^r(\ell)\big)}, \text{ else.}$$

Level game property, LG (Winter 1989). For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B \in \mathcal{B}^r$, $0 \leq r \leq h+1$, we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}).$$

[10]This axiom extends proportional balanced contributions (Besner 2016; Béal et al. 2017) and is related to balanced group contributions (Calvo, Lasaga and Winter 1996), an extension of balanced contributions (Myerson 1980).

[11]Winter (1989) called this axiom coalitional symmetry.

[12]These axioms extend proportionality in Besner (2017a).

4 The proportional Shapley hierarchy levels value

In Besner (2017b) the weighted Shapley levels value, presented in Gómez-Rúa and Vidal-Puga (2011), is extended to the weighted Shapley hierarchy levels values. It is not difficult to adapt these values to a proportional value for level structures. We have only to assign in the defining algorithm each weight of a coalition the original worth of the coalition.

Definition 4.1. *Let $N \in \mathcal{N}$ and $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}_{0+}^N$. Then the **proportional Shapley hierarchy levels value** Sh^{pHL} is defined by algorithm 4.1 below.*

Algorithm 4.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_{0+}^N$, $i \in N$, $\mathrm{w} \in \mathrm{W}$ such that $\mathrm{w}_S := v(S)$ for all $S \in \Omega^{B^h}$ and all $B^h \in \mathcal{B}^h$ if S is regarded as a player, Sh^{w} a weighted Shapley value and $(\mathcal{R}, \tilde{v}_i^r)$ TU-games, where $\mathcal{R}$ are nonempty, pairwise disjoint sets of some subsets $S \in \Omega^{\mathcal{B}^r(i)}$, $1 \leq r \leq h+1$, with $\tilde{v}_i^r(\mathcal{Q}) := v_i^r\big(\bigcup_{S \in \mathcal{Q}} S\big)$ for all $\mathcal{Q} \in \Omega^{\mathcal{R}}$, $v_i^r \in \mathcal{G}^{\mathcal{B}^r(i)}$. Take $v_i^{h+1} := v$.*

- ***Step k, $1 \leq k \leq h$:*** *Let $r := h - k + 1$. We define the TU-game $\big(\mathcal{B}^r(i), v_i^r\big)$ by*

$$v_i^r(T) := Sh_T^{\mathrm{w}}\Big(\big\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \neq \mathcal{B}^r(i)\big\} \cup \{T\}, \tilde{v}_i^{r+1}\Big) \text{ for all } T \in \Omega^{\mathcal{B}^r(i)}.$$

  In particular, $v_i^r\big(\mathcal{B}^r(i)\big)$ is the payoff assigned to component $\mathcal{B}^r(i)$.

- ***Step h+1:*** *The payoff Sh^{pHL} assigned to player i is given by*

$$Sh_i^{pHL}(N, v, \mathcal{B}) := Sh_{\{i\}}^{\mathrm{w}}\big(\{B \in \mathcal{B}^0 : B \subseteq \mathcal{B}^1(i)\}, \tilde{v}_i^1\big).$$

If $h = 0$, we only execute step $h + 1$.

Remark 4.2. *It is obvious, by construction, that Sh^{pHL} satisfies **E** and **LG**. If $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$ Sh^{pHL} coincides with Sh^p and so Sh^{pHL} is not additive too. It is easy to show, with a little example, that Sh^{pHL} doesn't satisfy **D**.*

For a fixed coalition function well-known facts of a weighted Shapley value hold also for the proportional Shapley value. We get similar results for the proportional Shapley hierarchy levels value and the weighted Shapley hierarchy levels values (Besner 2017b).

Corollary 4.3. *Let $N \in \mathcal{N}$ and $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}_{0+}^N$. Sh^{pHL} is the unique LS-value that satisfies **E** and **PBGC**.*

The proof is omitted because for a fixed coalition function we can replace the weights of the coalitions in a weighted Shapley hierarchy levels value by the worths of the coalitions and so we have a corollary to Besner (2017b, theorem 5.3).

5 The proportional Shapley support levels value

Just like the weighted Shapley support levels values (Besner 2017b), by the following value each player is "supported" by all components including her.

Definition 5.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$ and for all $T \subseteq N$, $T \ni i$,*

$$K_{v,T}(i) := \prod_{r=0}^{h} K_{v,T}^r(i), \text{ where} \tag{4}$$

$$K_{v,T}^r(i) := \frac{v\big(\mathcal{B}^r(i)\big)}{\sum_{\substack{B \in \mathcal{B}^r:\, B \subseteq \mathcal{B}^{r+1}(i),\\ B \cap T \neq \emptyset}} v(B)}. \tag{5}$$

The ***proportional Shapley support levels value*** *Sh^{pSL} is given by*

$$Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N,\, T \ni i} K_{v,T}(i) \Delta_v(T) \text{ for all } i \in N. \tag{6}$$

Obviously, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, Sh^{pSL} coincides with Sh^p. We present a set of axioms which are matched by this value.

Theorem 5.2. *The proportional Shapley support levels value Sh^{pSL} satisfies* ***E****,* ***D****,* ***L****,* ***H****,* ***WA****,* ***LG****,* ***SymBC****,* ***PBC****,* ***PWC*** *and* ***PTCC****.*

For the proof, see appendix 9.3. We would like to draw attention to the fact that also symmetry between components is satisfied. If a level structure is defined as above, so that the singletons are the elements of the lowest level, Winter (1989, remark 1.6) emphasized that the individual symmetry can be dropped for the standard axiomatization of the Shapley levels value. So the used axioms, efficiency, null player, symmetry between components and additivity, differ from the corresponding ones, satisfied by the proportional Shapley support levels value, only in the range (dummy instead of null player) and in the weaker form of additivity. Besner (2017a, theorem 5.3) axiomatized the proportional Shapley value by the efficiency, dummy, proportionality and weak additivity axioms for TU-values. If we extend these axioms to axioms for LS-values follows a "proportional" analogon to Winter's axiomatization.

Theorem 5.3. *Sh^{pSL} is the unique LS-value that satisfies* ***E****,* ***L****,* ***PBC*** *and* ***WA****.*

For the proof, see appendix 9.4.

5.1 Component substitution

One of the most remarkable properties of the proportional Shapley value is the satisfaction of the player splitting axiom (Besner 2017a). If a player splits in two new players which have in the sum the same effect to the other players as the splitted player before, in Besner (2017a) such a game is called a splitted player game, the payoff to uninvolved players doesn't change. We repeat the definition of a splitted player game.

Definition 5.4. *(Besner 2017a) Let $N, N^j \in \mathcal{N}$, $(N,v) \in \mathcal{G}^N$, $(N^j, v^j) \in \mathcal{G}^{N^j}$, $j \in N$, $k, \ell \in N^j$, $k, \ell \notin N$, $N^j := (N \backslash \{j\}) \cup \{k, \ell\}$. The game (N^j, v^j) is called a* ***splitted player game*** *to (N,v) if for all $S \subseteq N \backslash \{j\}$*

- $v^j(\{k\}) + v^j(\{\ell\}) = v(\{j\})$,

- $v^j(S \cup \{i\}) = v(S) + v^j(\{i\})$, $i \in \{k, \ell\}$,
- $v^j(S \cup \{k, l\}) = v(S \cup \{j\})$ *and*
- $v^j(S) = v(S)$.

By Banker (1981), an allocation scheme should not be sensitive to the way cost centres are organized. For splitted player games the following axiom satisfies Bankers demand.

Player splitting, PS. (Besner 2017a) For all $N \in \mathcal{N}$, $(N, v) \in \mathcal{G}^N$, $j \in N$ and a splitted player game $(N^j, v^j) \in \mathcal{G}^{N^j}$ to (N, v), we have

$$\varphi_i(N, v) = \varphi_i(N^j, v^j) \text{ for all } i \in N \backslash \{j\}.$$

This should hold also if the cost centres are structured hierarchically. We extend the definition of a splitted player game to LS-games.

Definition 5.5. *Let* N, $N^B \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $\underline{\mathcal{B}^B} = \underline{\mathcal{B}^B_h} \in \mathcal{L}^{N^B}$, $v \in \mathcal{GL}^N$, $v^B \in \mathcal{GL}^{N^B}$, $B \in \mathcal{B}^r$, $0 \leq r \leq h$, $B', B'' \subsetneq \mathfrak{U}$, $B' \cap (N \backslash B) = B'' \cap (N \backslash B) = B' \cap B'' = \emptyset$, $N^B = (N \backslash B) \cup B' \cup B''$, *such that*

$$(\mathcal{B}^B)^q = (\mathcal{B}^q \backslash \{B^q\}) \cup \{(B^B)^q\},\ B \subseteq B^q \in \mathcal{B}^q,\ (B^B)^q = (B^q \backslash B) \cup B' \cup B''$$

*in the q-th level of $\mathcal{B}^B$, $r \leq q \leq h+1$. The game $(N^B, v^B, \underline{\mathcal{B}^B}) \in \mathcal{GL}^{N^B}$ is called a **component substitution game (CS-game)** to $(N, v, \underline{\mathcal{B}})$ if for all S,* $S = \bigcup_{\substack{B^r \subseteq S, \\ B^r \in \mathcal{B}^r}} B^r$, $S \cap B = \emptyset$,

- $v^B(B') + v^B(B'') = v(B)$,
- $v^B(S \cup B' \cup B'') = v(S \cup B)$,
- $v^B(S) = v(S)$ *and*
- $v^B(S \cup C) = v(S) + v^B(C)$, $C \in \{B', B''\}$.

It is evident, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, that a CS-game coincides with a splitted player game.

Remark 5.6. *In the CS-game the worth of all coalitions which have not only components of the r-th level as subsets is arbitrary within the domain. The structure of the components of all levels lower then r must only be conform with the definition of a level structure.*

Remark 5.7. *In the induced r-th level game of the CS-game the substituting components are weakly dependent. If the substituted component is a singleton and is replaced by singletons, the players of the substituting singletons are weakly dependent in the CS-game.*

The r-th level game of the CS-game is a splitted player game to the original r-th level game if we regard both games as TU-games. In the following axiom the payoff to members of uninvolved components doesn't change in the sum in a CS-game in relation to the original game.

Component substitution, CS. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B \in \mathcal{B}^r$, $0 \leq r \leq h$, and $(N^B, v^B, \underline{\mathcal{B}^B}) \in \mathcal{GL}^{N^B}$ a CS-game to $(N, v, \underline{\mathcal{B}})$, we have

$$\sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B^r} \varphi_i(N^B, v^B, \underline{\mathcal{B}^B}) \text{ for all } B^r \in (\mathcal{B}^r \backslash \{B\}).$$

It is clear, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, that **CS** corresponds to **PS**.

Remark 5.8. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B \in \mathcal{B}^r$, $0 \leq r \leq h$, and $(N^B, v^B, \underline{\mathcal{B}^B}) \in \mathcal{GL}^{N^B}$ a CS-game to $(N, v, \underline{\mathcal{B}})$ with related components B' and B''. If φ is an LS-value that satisfies **E** and **CS**, then we have*

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B'} \varphi_i(N^B, v^B, \underline{\mathcal{B}^B}) + \sum_{i \in B''} \varphi_i(N^B, v^B, \underline{\mathcal{B}^B}).$$

It turns out that the proportional Shapley support levels value is doing well with **CS**.

Proposition 5.9. *Sh^{pSL} satisfies **CS**.*

For the proof, see appendix 9.5. The following lemmas show dependence on **SymBC** and **PBC**, respectively, for efficient LS-values which satisfy **CS**.

Lemma 5.10. *Let $N \in \mathcal{N}$ and $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}_0^N$. If an LS-value φ satisfies **E** and **CS** then φ satisfies also **SymBC**.*

For the proof, see appendix 9.6.

Lemma 5.11. *Let $N \in \mathcal{N}$ and $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}_{0_{\mathbb{Q}}}^N$. If an LS-value φ satisfies **E** and **CS** then φ satisfies also **PBC**.*

For the proof, see appendix 9.7. We extend corollary 5.13 in Besner (2017a).

Corollary 5.12. *Let $N \in \mathcal{N}$ and $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}_{0_{\mathbb{Q}}}^N$. Sh^{pSL} is the unique LS-value that satisfies **E**, **L**, **CS** and **WA**.*

Obviously follows the proof by proposition 5.9 and lemma 5.11 from theorem 5.3.

Remark 5.13. *Lemma 5.11 holds for $v \in \mathcal{LG}_0^N$ if we require continuity of the LS-value in $v(B)$ for all $v \in \mathcal{LG}_0^N$ and all $B \in \overline{\mathcal{B}}$ in an additional axiom. So also corollary 5.12 is valid for $v \in \mathcal{LG}_0^N$ if there is required an additional continuity axiom.*

6 The proportional Shapley alliance levels value

The following value allows a greater independence for subgroups of a component from the nesting components.

Definition 6.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_{0+}^N$ and for all $T \subseteq N$, $T \ni i$,*

$$A_{v,T}(i) := \prod_{r=0}^{h} A_{v,T}^r(i), \text{ where}$$
$$A_{v,T}^r(i) := \frac{v\big(\mathcal{B}^r(i) \cap T\big)}{\sum_{\substack{B \in \mathcal{B}^r:\, B \subseteq \mathcal{B}^{r+1}(i),\\ B \cap T \neq \emptyset}} v(B \cap T)}.$$

*The **proportional Shapley alliance levels value** Sh^{pAL} is given by*

$$Sh_i^{pAL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N,\, T \ni i} A_{v,T}(i) \Delta_v(T) \text{ for all } i \in N. \tag{7}$$

It is evident, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, that Sh^{pAL} coincides with Sh^p.

Theorem 6.2. *The proportional Shapley alliance levels value Sh^{pAL} satisfies **E**, **D**, **L**, **H**, **DO**, **LO**, **CWA**, **PWC** and **PTCC**.*

For the proof, see appendix 9.8. If a value satisfies loyal player out, the loyal player has no influence to the payoff of players which are not in the same top component as the loyal player. If a player i is in the same top component as the loyal player, there is no difference in the impact of the loyal player to the payoff of player i in the original game and the restriction of the cut $\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big)$. The next axiomatization uses loyal player out.

Theorem 6.3. *Sh^{pAL} is the unique LS-value that satisfies **E**, **LO**, **PTCC** and **CWA**.*

For the proof, see appendix 9.9.

7 The proportional Shapley collaboration levels value

Our last value allows each component to act independently from the including component.

Definition 7.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$ and for all $T \subseteq N$, $T \ni i$,*

$$C_{v,T}(i) := \prod_{r=0}^{h} C_{v,T}^r(i), \text{ with}$$
$$C_{v,T}^r(i) := \frac{v(\mathcal{B}_T^r(i))}{\sum_{B \in \widehat{\mathcal{B}_T^{r+1}(i)}} v(B)},$$

where $\mathcal{B}_T^r(i)$ is the largest component $\mathcal{B}^\ell(i)$, $0 \leq \ell \leq r$, with $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}_T^{h+1}(i) := T$ and

$$\widehat{\mathcal{B}_T^{r+1}(i)} := \begin{cases} \mathcal{B}_T^r(i), & \text{if } \mathcal{B}_T^r(i) = \mathcal{B}_T^{r+1}(i), \\ \big\{B \in \overline{\mathcal{B}} :\ B \subsetneq \mathcal{B}_T^{r+1}(i),\ B \nsubseteq B' \in \overline{\mathcal{B}},\ B' \subsetneq \mathcal{B}_T^{r+1}(i)\big\}, & \text{else,} \end{cases}$$

is the set of all largest components which are subsets of $\mathcal{B}_T^{r+1}(i)$.

*The **proportional Shapley collaboration levels value** Sh^{pCL} is given by*

$$Sh_i^{pCL}(N, v, \underline{\mathcal{B}}) = \sum_{T \subseteq N,\, T \ni i} C_{v,T}(i) \Delta_v(T) \text{ for all } i \in N.$$

Obviously, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, Sh^{pCL} coincides with Sh^p.

Theorem 7.2. *The proportional Shapley collaboration levels value Sh^{pCL} satisfies **E**, **D**, **L**, **H**, **IDO**, **ILO**, **WA**, **PWC** and **PTCC**.*

The proof is omitted because it is analogous to the proof of theorem 6.2 or rather theorem 5.2. The following axiomatization uses internal loyal player out.

Theorem 7.3. *Sh^{pCL} is the unique LS-value that satisfies **E**, **ILO**, **PWC** and **WA**.*

For the proof, see appendix 9.10.

8 Conclusion

If $\underline{\mathcal{B}} = \underline{\mathcal{B}_1}$, we get, with a simple reformulation of our values, also four new proportional coalitional values for coalition structures with related axiomatizations, the proportional Shapley hierarchy coalitional value, the proportional Shapley support coalitional value, the proportional Shapley alliance coalitional value and the proportional Shapley collaboration coalitional value.

The proportional Shapley hierarchy levels value doesn't satisfy the dummy axiom. This means that a dummy player can get a payoff which is higher as her singleton worth. Therefore players must obtain also shares of dividends from coalitions which do not contain these players. In many situations it seems naturally that if a component supports subsets of the component we should admit that the shares of dummy players,besides their singleton worth, are not so null because the dummy players affect the supporting weight, here the worth, of the component.[13]

The other three values satisfy the dummy player axiom. By the proportional Shapley support levels value players of coalitions are also supported by possible dummy players within components containing them. But the dummy players don't receive a reward for there assistance and it is easy to proof that this value doesn't satisfy the dummy player out property. Whereas for the weighted Shapley support levels values the weights of components in restricted level structures, where a null player is removed, are not automatically defined, here the worths of the restricted components are given. The restricted components normally have a reduced worth and so a reduced weight for the proportional Shapley support levels value. In contrary, looking at the weighted Shapley support levels values, we can define a new weighted Shapley support levels value for the game with the restricted level structure where the restricted components get the same weights as the complete components in the original game with the unrestricted level structure. This is especially true for the special case of the Shapley levels value where all components have the same weight.

Our two last values, the proportional Shapley alliance levels value and the proportional Shapley collaboration levels value, avoid this lack of receiving a reward for dummy players because there is no support of dummy players to any other coalitions. This is confirmed by the dummy player out properties of these values.

The extension of the property of player splitting to level structures was only examined for the proportional Shapley support levels value. For this value we found that the

[13]For more details see also for the role of null players in Gómez-Rúa and Vidal-Puga (2010).

component substitution axiom is satisfied which allows an interesting axiomatization as a corollary and gives a main difference, besides the proportionality between components axiom, to the Shapley levels value. The introduction of extensions of the player splitting axiom for the other LS-values is left to further research.

9 Appendix

Convention 9.1. To avoid cumbersome case distinctions in the proves using **PBC**, **PWC** or **PTCC**, if there is only one single player assessed in isolation she is defined in this cases as weakly dependent (rc-dependent, tc-dependent) by herself. Then **PBC**, **PWC** or **PTCC** can be used and are trivially satisfied.

9.1 Proof of lemma 3.2

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $i \in N$, $S \subseteq \mathcal{B}^h(i)\backslash\{i\}$, and $T \subseteq N\backslash\mathcal{B}^h(i)$. Then exists an unique $R \subseteq N$, $R \nsubseteq \mathcal{B}^h(i)$, $R \ni i$ with $R = S \cup T \cup \{i\}$ and for each $R \subseteq N$, $R \nsubseteq \mathcal{B}^h(i)$, $R \ni i$, exists an unique S and T above with $R = S \cup T \cup \{i\}$. If $T = \emptyset$ eq. (3) is trivially satisfied.

Let $t := |T|$, $t \geq 1$. We show by an induction I_1 on the size $s := |S|$

$$v(S \cup T \cup \{i\}) - v(S \cup T) = v(S \cup \{i\}) - v(S) \Leftrightarrow \Delta_v(S \cup T \cup \{i\}) = 0.$$

Initialisation I_1: Let $S := \emptyset$ and so $s = 0$. We use a second induction I_2 on the size t.
Initialisation I_2: Let $t = 1$. We get

$$\begin{array}{rrcl} & v(S \cup T \cup \{i\}) - v(S \cup T) & = & v(S \cup \{i\}) - v(S) \\ \Leftrightarrow & v(T \cup \{i\}) - v(T) & = & v(\{i\}) \\ \underset{(1)}{\Leftrightarrow} & \sum_{Q \subseteq (T \cup \{i\})} \Delta_v(Q) - \sum_{Q \subseteq T} \Delta_v(Q) & = & \Delta_v(\{i\}) \\ \Leftrightarrow & \sum_{Q \subseteq (T \cup \{i\}), Q \ni i} \Delta_v(Q) & = & \Delta_v(\{i\}) \\ \Leftrightarrow & \Delta_v(S \cup T \cup \{i\}) & = & 0. \end{array}$$

Induction step I_2: Assume that equality and such equivalence in the first and last line of the system above hold for all coalitions $\tilde{T} \in \Omega^{N\backslash\mathcal{B}^h(i)}$, $|\tilde{T}| \leq t'$, $t' \geq 1$, (IH_2) and let $t = t' + 1$. It follows

$$\begin{array}{rrcl} & v(S \cup T \cup \{i\}) - v(S \cup T) & = & v(S \cup \{i\}) - v(S) \\ \Leftrightarrow & v(T \cup \{i\}) - v(T) & = & v(\{i\}) \\ \underset{(1)}{\Leftrightarrow} & \sum_{Q \subseteq (T \cup \{i\})} \Delta_v(Q) - \sum_{Q \subseteq T} \Delta_v(Q) & = & \Delta_v(\{i\}) \\ \Leftrightarrow & \sum_{Q \subseteq (T \cup \{i\}), Q \ni i} \Delta_v(Q) & = & \Delta_v(\{i\}) \\ \underset{(IH_2)}{\Leftrightarrow} & \Delta_v(S \cup T \cup \{i\}) & = & 0. \end{array}$$

Induction step I_1: Assume that equality and such equivalence in the first and last line of the system above hold for all coalitions $\tilde{S} \subseteq \mathcal{B}^h(i) \backslash \{i\}$, $|\tilde{S}| \leq s'$, $s' \geq 0$, (IH_1) and let $s = s' + 1$. We get

$$\begin{array}{rrcl} & v(S \cup T \cup \{i\}) - v(S \cup T) & = & v(S \cup \{i\}) - v(S) \\ \underset{(1)}{\Leftrightarrow} & \sum\limits_{Q \subseteq (S \cup T \cup \{i\})} \Delta_v(Q) - \sum\limits_{Q \subseteq S \cup T} \Delta_v(Q) & = & \sum\limits_{Q \subseteq (S \cup \{i\})} \Delta_v(Q) - \sum\limits_{Q \subseteq S} \Delta_v(Q) \\ \Leftrightarrow & \sum\limits_{Q \subseteq (S \cup T \cup \{i\}), Q \ni i} \Delta_v(Q) & = & \sum\limits_{Q \subseteq (S \cup \{i\}), Q \ni i} \Delta_v(Q) \\ \underset{(IH_1)}{\Leftrightarrow} & \Delta_v(S \cup T \cup \{i\}) & = & 0. \end{array}$$

□

9.2 Proof of lemma 3.4

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $0 \leq r \leq h$, $v \in \mathcal{GL}^N$, $i, j \in N$, $i \neq j$, $S \in \Omega^{N \backslash \{i,j\}}$, $S \nsubseteq \mathcal{B}^r(i)$, and $T := \big(\mathcal{B}^r(i) \cap S\big)$. We show by induction on the size $s := |S|$, $s \geq 1$,

$$v(S \cup \{i\}) = v(S) + v(T \cup \{i\}) - v(T) \quad \Leftrightarrow \quad \Delta_v(S \cup \{i\}) = 0.$$

Initialisation: Let $s = 1$. We have $T = \emptyset$ and get

$$\begin{array}{rrcl} & v(S \cup \{i\}) & = & v(S) + v(T \cup \{i\}) - v(T) \\ \Leftrightarrow & v(S \cup \{i\}) & = & v(S) + v(\{i\}) \\ \underset{(1)}{\Leftrightarrow} & \Delta_v(S \cup \{i\}) + \Delta_v(S) + \Delta_v(\{i\}) & = & \Delta_v(S) + \Delta_v(\{i\}) \\ \Leftrightarrow & \Delta_v(S \cup \{i\}) & = & 0. \end{array}$$

Induction step: Assume that equivalence and equality in the first and last line of the system above hold for all coalitions $\tilde{S} \in \Omega^{N \backslash \{i,j\}}$, $|\tilde{S}| \leq s'$, $s' \geq 1$, (IH) and let $s = s' + 1$. We have $T \subsetneq S$ and get

$$\begin{aligned} v(T \cup \{i\}) - v(T) &\underset{(1)}{=} \sum_{R \subseteq (T \cup \{i\})} \Delta_v(R) - \sum_{R \subseteq T} \Delta_v(R) \\ &= \sum_{R \subseteq T} \Delta_v(R) + \sum_{R \subseteq (T \cup \{i\}),\, R \ni i} \Delta_v(R) - \sum_{R \subseteq T} \Delta_v(R) \underset{(IH)}{=} \Delta_v(\{i\}). \end{aligned} \tag{8}$$

It follows

$$\begin{array}{rrcl} & v(S \cup \{i\}) & = & v(S) + v(T \cup \{i\}) - v(T) \\ \underset{(1),(8)}{\Leftrightarrow} & \Delta_v(S \cup \{i\}) + \sum\limits_{R \subsetneq (S \cup \{i\})} \Delta_v(R) & = & \sum\limits_{R \subseteq S} \Delta_v(R) + \Delta_v(\{i\}) \\ \underset{(IH)}{\Leftrightarrow} & \Delta_v(S \cup \{i\}) + \Delta_v(\{i\}) + \sum\limits_{R \subseteq S} \Delta_v(R) & = & \sum\limits_{R \subseteq S} \Delta_v(R) + \Delta_v(\{i\}) \\ \Leftrightarrow & \Delta_v(S \cup \{i\}) & = & 0. \end{array}$$

Analogous with player j. □

Lemma 9.2 (Besner 2017b). *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$. Each $S \in \Omega^N$ is a subset of exactly one coalition $T^r \in \Omega^N$, $T^r = \bigcup_{\substack{B^r \subseteq T^r,\, B^r \in \mathcal{B}^r,\\ B^r \cap S \neq \emptyset}} B^r$. Thus each $S \in \Omega^N$ is also uniquely referred to as S_{T^r}.*

Lemma 9.3 (Besner 2017b). *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$, and S_{T^r} the coalitions from lemma 9.2 with related coalitions T^r. Then we have in the induced game $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)$ for each $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, related to $T^r \in \Omega^N$,*

$$\Delta_{v^r}(\mathcal{T}^r) = \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}).$$

Lemma 9.4 (Besner 2017a). *Players $i, j \in N$ are weakly dependent in v, $v \in \mathcal{G}^N$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \in \Omega^{N \backslash \{i,j\}}$.*

Remark 9.5. *We can consider the collection of all LS-games $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$, $N \in \mathcal{N}$, as a vector space $\mathbb{R}^{2^n-1}$. Each game v is represented by a vector $\overrightarrow{v} \in \mathbb{R}^{2^n-1}$, where the entries in the $2^n - 1$ coordinates of the $2^n - 1$ coalitions $S \in \Omega^N$ get the worth $v(S)$ of the respective coalition S. Hence there exists for every game v a vector $\overrightarrow{\Delta_v} \in \mathbb{R}^{2^n-1}$, which is corresponding to the vector $\overrightarrow{v}$, where the entries of the coordinates get the dividends of the respective coalitions. By (1) we get with $v, v_1, v_2 \in \mathcal{G}^N$*

$$\begin{aligned}
& \overrightarrow{\Delta_v} = \overrightarrow{\Delta_{v_1}} + \overrightarrow{\Delta_{v_2}} \\
\Leftrightarrow \quad & \Delta_v(S) = \Delta_{v_1}(S) + \Delta_{v_2}(S) \text{ for all } S \subseteq N \\
\Leftrightarrow \quad & v(S) - \sum_{R \subsetneq S} \Delta_v(R) = v_1(S) - \sum_{R \subsetneq S} \Delta_{v_1}(R) + v_2(S) - \sum_{R \subsetneq S} \Delta_{v_2}(R) \text{ for all } S \subseteq N \\
\Leftrightarrow \quad & v = v_1 + v_2.
\end{aligned}$$

Lemma 9.6. ***E** and **LO** imply **L**.*

Proof. Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$ and φ an LS-value that satisfies **E** and **LO**. It is well-known for a TU-value that efficiency and dummy player out imply dummy. So lemma 9.6 is clear if $h = 0$. Let now $h \geq 1$. We get for a loyal player $j \in N$

$$\begin{aligned}
\varphi_j(N, v, \underline{\mathcal{B}}) &\underset{\textbf{(E)}}{=} v(N) - \sum_{i \in N \backslash (\{j\})} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{\textbf{(LO)}}{=} v(N) - \sum_{i \in N \backslash (\{j\})} \varphi_i\big(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}}\big) \\
&\quad - \sum_{i \in \mathcal{B}^h(j) \backslash \{j\}} \Big[\varphi_i\big(\mathcal{B}^h(j), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(j)}\big) - \varphi_i\big(\mathcal{B}^h(j) \backslash \{j\}, v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(j) \backslash \{j\}}\big)\Big] \\
&\underset{\textbf{(E)}}{=} v(N) - v(N \backslash \{j\}) - \Big[v\big(\mathcal{B}^h(j)\big) - \varphi_j\big(\mathcal{B}^h(j), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(j)}\big) - v\big(\mathcal{B}^h(j) \backslash \{j\}\big)\Big] \\
&\underset{(1)}{=} \sum_{S \subseteq N} \Delta_v(S) - \sum_{S \subseteq N \backslash \{j\}} \Delta_v(S) - \sum_{S \subseteq \mathcal{B}^h(j)} \Delta_v(S) + \sum_{S \subseteq \mathcal{B}^h(j) \backslash \{j\}} \Delta_v(S) + \varphi_j\big(\mathcal{B}^h(j), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(j)}\big) \\
&\underset{\text{Lem. } 3.2}{=} \varphi_j\big(\mathcal{B}^h(j), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(j)}\big).
\end{aligned}$$

□

Lemma 9.7. ***E** and **ILO** imply **L**.*

The proof is omitted because it is completely analogous to the proof of lemma 9.6.

9.3 Proof of theorem 5.2

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}_0^N$ and $K_{v,T}$ and $K_{v,T}^r$ the expressions according to def. 5.1.

- **E**: Let $T \subseteq N$, $T \ni i$, $0 \leq r \leq h$. Obviously we have

$$\sum_{j \in \mathcal{B}^{r+1}(i),\, j \in T} \prod_{\ell=0}^{r} K_{v,T}^r(j) = 1.$$

Thus $\sum_{i \in T} K_{v,T}(i) = 1$ and it follows

$$\sum_{i \in N} Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) = \sum_{i \in N} \sum_{\substack{T \subseteq N, \\ T \ni i}} K_{v,T}(i)\Delta_v(T) = \sum_{T \subseteq N} \sum_{i \in T} K_{v,T}(i)\Delta_v(T) = \sum_{T \subseteq N} \Delta_v(T) = v(N)$$

and **E** is shown.

- **D**: It is well-known fact that a player $i \in N$ is a dummy player iff $\Delta_v(S \cup \{i\}) = 0$ for all $S \in \Omega^{N \setminus \{i\}}$. So by eq. (6) it's obvious that **D** is satisfied.
- **L**: It's obvious, by lemma 3.2 and eq. (6), that **L** is satisfied.
- **H**: It is well-known or easy to check by (1) that dividends are homogeneous, $\Delta_{\alpha v}(S) = \alpha \Delta_v(S)$ for all $S \subseteq N$, $\alpha \in \mathbb{R}$. So we have

$$\begin{aligned} Sh_i^{pSL}(N, \alpha v, \underline{\mathcal{B}}) &\underset{(6)}{=} \sum_{T \subseteq N, T \ni i} K_{\alpha v,T}(i)\Delta_{\alpha v}(T) = \alpha \sum_{T \subseteq N, T \ni i} K_{v,T}(i)\Delta_v(T) \\ &= \alpha Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) \text{ for all } i \in N,\ \alpha > 0. \end{aligned}$$

- **WA**: Let $v'(B) = c \cdot v(B)$ for all $B \in \overline{\mathcal{B}}$, $c > 0$. It follows $K_{v,T}^r = K_{v',T}^r$ and thus $K_{v,T} = K_{v',T}$. So we have for all $i \in N$

$$\begin{aligned} Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) + Sh_i^{pSL}(N, v', \underline{\mathcal{B}}) &\underset{(6)}{=} \sum_{T \subseteq N, T \ni i} K_{v,T}(i)\Delta_v(T) + \sum_{T \subseteq N, T \ni i} K_{v',T}(i)\Delta_{v'}(T) \\ &= \sum_{T \subseteq N, T \ni i} K_{v,T}(i)\big[\Delta_v(T) + \Delta_{v'}(T)\big] \underset{\substack{Rem. \\ 9.5}}{=} Sh_i^{pSL}(N, v + v', \underline{\mathcal{B}}). \end{aligned}$$

- **LG**: Let $B^r \in \mathcal{B}^r$, $0 \leq r \leq h+1$. If $r = 0$, **LG** trivially is satisfied because the 0-th level game corresponds to the original LS-game, if $r = h+1$, **LG** is satisfied by **E**.

Let now $1 \leq r \leq h$. We have for all $S \subseteq N$, $S \cap B^r \neq \emptyset$,

$$\sum_{i \in B^r, i \in S} \prod_{\ell=0}^{r-1} K_{v,S}^{\ell}(i) = 1. \tag{9}$$

In the game $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})$ we have for all $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, $B^r \in \mathcal{T}^r$,

$$K_{v^r,\mathcal{T}^r}(B^r) = \prod_{\ell=r}^{h} K_{v^r,\mathcal{T}^r}^{\ell-r}(B^r). \tag{10}$$

Let $i \in B^r$, $r \le \ell \le h$ and S_{T^r} the coalitions from lemma 9.2. We have $\mathcal{B}^\ell(i) = \mathcal{B}^\ell(B^r)$. Notice that for each $\mathcal{T}^r \subseteq \mathcal{B}^r$, related to $T^r \in \Omega^N$, if $i \in S_{T^r}$ also $B^r \in \mathcal{T}^r$. It follows for all $S_{T^r} \in \Omega^N$, $i \in S_{T^r}$,

$$\begin{aligned} K^\ell_{v,S_{T^r}}(i) &\underset{(5)}{=} \frac{v\big(\mathcal{B}^\ell(i)\big)}{\sum_{\substack{B\in\mathcal{B}^\ell: B\subseteq\mathcal{B}^{\ell+1}(i),\\ B\cap S_{T^r}\neq\emptyset}} v(B)} \underset{\substack{Lem.\\ 9.2}}{=} \frac{v\big(\mathcal{B}^\ell(B^r)\big)}{\sum_{\substack{B\in\mathcal{B}^\ell: B\subseteq\mathcal{B}^{\ell+1}(B^r),\\ B\cap T^r\neq\emptyset}} v(B)} \\ &\underset{(2)}{=} \frac{v^r\big({\mathcal{B}^r}^{\ell-r}(B^r)\big)}{\sum_{\substack{B\in{\mathcal{B}^r}^{\ell-r}: B\subseteq{\mathcal{B}^r}^{\ell+1-r}(B^r),\\ B\cap\mathcal{T}^r\neq\emptyset}} v^r(B)} \underset{(5)}{=} K^{\ell-r}_{v^r,\mathcal{T}^r}(B^r). \end{aligned} \tag{11}$$

Thus we have for all $S_{T^r} \in \Omega^N$, $B^r \in \mathcal{T}^r$, $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$,

$$\begin{aligned} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} K_{v,S_{T^r}}(i) &\underset{(4)}{=} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} \prod_{\ell=0}^{h} K^\ell_{v,S_{T^r}}(i) \underset{(11)}{=} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} \prod_{\ell=0}^{r-1} K^\ell_{v,S_{T^r}}(i) \prod_{\ell=r}^{h} K^{\ell-r}_{v^r,\mathcal{T}^r}(B^r) \\ &\underset{(9)}{=} \prod_{\ell=r}^{h} K^{\ell-r}_{v^r,\mathcal{T}^r}(B^r) \underset{(10)}{=} K_{v^r,\mathcal{T}^r}(B^r). \end{aligned} \tag{12}$$

Finally we get

$$\begin{aligned} \sum_{i\in B^r} Sh^{pSL}_i(N,v,\underline{\mathcal{B}}) &\underset{(6)}{=} \sum_{i\in B^r}\sum_{\substack{S\subseteq N,\\ S\ni i}} K_{v,S}(i)\Delta_v(S) \underset{\substack{Lem.\\ 9.2}}{=} \sum_{i\in B^r}\sum_{\substack{S_{T^r}\subseteq N,\\ S_{T^r}\ni i}} K_{v,S_{T^r}}(i)\Delta_v(S_{T^r}) \\ &= \sum_{S_{T^r}\subseteq N}\sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} K_{v,S_{T^r}}(i)\Delta_v(S_{T^r}) \underset{(12)}{=} \sum_{\substack{S_{T^r}\subseteq N,\\ \mathcal{T}^r\ni B^r}} K_{v^r,\mathcal{T}^r}(B^r)\Delta_v(S_{T^r}) \\ &\underset{\substack{Lem.\\ 9.2}}{=} \sum_{\mathcal{T}^r\subseteq\mathcal{B}^r,\,\mathcal{T}^r\ni B^r} K_{v^r,\mathcal{T}^r}(B^r) \sum_{S_{T^r}\subseteq T^r} \Delta_v(S_{T^r}) \\ &\underset{\substack{Lem.\\ 9.3}}{=} \sum_{\mathcal{T}^r\subseteq\mathcal{B}^r,\,\mathcal{T}^r\ni B^r} K_{v^r,\mathcal{T}^r}(B^r)\Delta_{v^r}(\mathcal{T}^r) = Sh^{pSL}_{B^r}(\mathcal{B}^r,v^r,\underline{\mathcal{B}^r}). \end{aligned}$$

• **SymBC**: We have, by (2), $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}_0^{\mathcal{B}^r}$, $0 \le r \le h$. Let $k, \ell \in N$, $0 \le r \le h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$ symmetric in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})$. If $r = 0$ we have, as a well-known fact, $\Delta_v(S \cup \{k\}) = \Delta_v(S \cup \{\ell\})$ for all $S \subseteq N \backslash \{k, \ell\}$ and get

$$\begin{aligned} Sh^{pSL}_k(N,v,\underline{\mathcal{B}}) &\underset{(6)}{=} \sum_{T\subseteq N,\,T\ni k} K_{v,T}(k)\Delta_v(T) \\ &= \sum_{S\subseteq N\backslash\{k,\ell\}} K_{v,S\cup k}(k)\Delta_v(S\cup k) + \sum_{T\subseteq N,\,\{k,\ell\}\subseteq T} K_{v,T}(k)\Delta_v(T) \\ &\underset{\substack{Def.\\ 5.1}}{=} \sum_{S\subseteq N\backslash\{k,\ell\}} K_{v,S\cup\ell}(\ell)\Delta_v(S\cup\ell) + \sum_{T\subseteq N,\,\{k,\ell\}\subseteq T} K_{v,T}(\ell)\Delta_v(T) = Sh^{pSL}_\ell(N,v,\underline{\mathcal{B}}). \end{aligned}$$

Thus we have also in the r-th level game, $0 \le r \le h$,

$$Sh^{pSL}_{\mathcal{B}^r(k)}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) = Sh^{pSL}_{\mathcal{B}^r(\ell)}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})$$

and the claim follows by **LG**.

• **PBC**: We have, by (2), $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}_0^{\mathcal{B}^r}$, $0 \le r \le h$. Let $k, \ell \in N$, $0 \le r \le h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$ weakly dependent in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})$. If $r = 0$ we get

$$\frac{Sh_k^{pSL}(N,v,\underline{\mathcal{B}})}{v(\{k\})} \underset{(6)}{=} \sum_{T\subseteq N,\, T\ni k} \frac{K_{v,T}(k)}{v(\{k\})}\Delta_v(T) \underset{\substack{Lem.\\ 9.4}}{=} K_{v,\{k\}}(k) + \sum_{T\subseteq N,\, \{k,\ell\}\subseteq T} \frac{K_{v,T}(k)}{v(\{k\})}\Delta_v(T)$$

$$\underset{\substack{Def.\\ 5.1}}{=} K_{v,\{\ell\}}(\ell) + \sum_{T\subseteq N,\, \{k,\ell\}\subseteq T} \frac{K_{v,T}(\ell)}{v(\{\ell\})}\Delta_v(T) = \sum_{T\subseteq N,\, T\ni \ell} \frac{K_{v,T}(\ell)}{v(\{\ell\})}\Delta_v(T) = \frac{Sh_\ell^{pSL}(N,v,\underline{\mathcal{B}})}{v(\{\ell\})}.$$

Thus we have also in the r-th level game, $0 \le r \le h$,

$$\frac{Sh_{\mathcal{B}^r(k)}^{pSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})}{v^r\big(\{\mathcal{B}^r(k)\}\big)} = \frac{Sh_{\mathcal{B}^r(\ell)}^{pSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})}{v^r\big(\{\mathcal{B}^r(\ell)\}\big)}$$

and the claim follows by **LG** and (2).

• **PWC**: Let $k, \ell \in N$, $0 \le r \le h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and all $i \in \mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)$ rc-dependent in v. We get

$$\sum_{i\in\mathcal{B}^r(k)} \frac{Sh_i^{pSL}(N,v,\underline{\mathcal{B}})}{v\big(\mathcal{B}^r(k)\big)} \underset{\substack{Def.\\ 5.1}}{=} \frac{1}{v\big(\mathcal{B}^r(k)\big)} \sum_{i\in\mathcal{B}^r(k)} \sum_{T\subseteq N,\, T\ni i} \left[\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} v(B)}\right]\Delta_v(T)$$

$$\begin{aligned}
&\underset{\substack{Lem.\\ 3.4}}{=} \frac{1}{v\big(\mathcal{B}^r(k)\big)} \sum_{i\in\mathcal{B}^r(k)} \left[\sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \left[\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} v(B)}\right]\Delta_v(T) \right.\\
&\qquad \left. + \sum_{\substack{S\subseteq\mathcal{B}^r(k),\\ S\ni i}} \left[\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap S\neq\emptyset}} v(B)}\right]\Delta_v(S)\right]\\
&= \frac{1}{v\big(\mathcal{B}^r(k)\big)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \sum_{i\in\mathcal{B}^r(k)} \left[\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} v(B)}\right]\Delta_v(T)\\
&\qquad + \frac{1}{v\big(\mathcal{B}^r(k)\big)} \sum_{S\subseteq\mathcal{B}^r(k)} \left[\sum_{i\in S}\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap S\neq\emptyset}} v(B)}\right]\Delta_v(S)\\
&= \frac{1}{v\big(\mathcal{B}^r(k)\big)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \left[\prod_{j=r}^{h} \frac{v\big(\mathcal{B}^j(k)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} v(B)}\right.\\
&\qquad \left.\cdot \sum_{i\in\mathcal{B}^r(k)} \prod_{j=0}^{r-1} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} v(B)} {}^{14}\right] + \frac{\sum_{S\subseteq\mathcal{B}^r(k)}\Delta_v(S)}{v\big(\mathcal{B}^r(k)\big)}\\
&\underset{(1)}{=} \frac{1}{v\big(\mathcal{B}^r(k)\big)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r}^{h} \frac{v\big(\mathcal{B}^j(k)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} v(B)} + 1\\
&= \frac{1}{v\big(\mathcal{B}^r(\ell)\big)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r}^{h} \frac{v\big(\mathcal{B}^j(\ell)\big)}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(\ell),\\ B\cap T\neq\emptyset}} v(B)} + 1 = \sum_{i\in\mathcal{B}^r(\ell)} \frac{Sh_i^{pSL}(N,v,\underline{\mathcal{B}})}{v\big(\mathcal{B}^r(\ell)\big)}.
\end{aligned}$$

• **PTCC**: Let $k, \ell \in N$, $0 \leq r \leq h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and all $i \in \mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)$ are tc-dependent in v. If $h = 0$, Sh^{pSL} satisfies **PTCC** because Sh^{pSL} coincides with Sh^p, **PTCC** coincides with proportionality and Sh^p satisfies proportionality (see theorem 5.3 in Besner (2017a)).

Let now $h \geq 1$. We get

$$\sum_{i \in B^r(k)} \frac{Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) - Sh_i^{pSL}\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big)}{v\big(B^r(k)\big)}$$

$$\underset{\substack{Def.\\ 5.1}}{=} \frac{1}{v\big(B^r(k)\big)} \sum_{i \in B^r(k)} \left[\sum_{T \subseteq N, T \ni i} K_{v,T}(i)\Delta_v(T) - \sum_{T \subseteq \mathcal{B}^h(i), T \ni i} K_{v,T}(i)\Delta_v(T) \right]$$

$$\underset{\substack{Lem.\\ 3.4}}{=} \frac{1}{v\big(B^r(k)\big)} \sum_{i \in B^r(k)} \sum_{\substack{T \subseteq N, T \not\subseteq \mathcal{B}^h(i),\\ (\mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)) \subseteq T}} K_{v,T}(i)\Delta_v(T)$$

$$\underset{\substack{Def.\\ 5.1}}{=} \frac{1}{v\big(B^r(k)\big)} \sum_{i \in B^r(k)} \sum_{\substack{T \subseteq N, T \not\subseteq \mathcal{B}^h(i),\\ (\mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)) \subseteq T}} \left[\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B \in \mathcal{B}^j:\, B \subseteq \mathcal{B}^{j+1}(i),\\ B \cap T \neq \emptyset}} v(B)} \right] \Delta_v(T)$$

$$= \frac{1}{v\big(B^r(k)\big)} \sum_{\substack{T \subseteq N, T \not\subseteq \mathcal{B}^h(k),\\ (\mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)) \subseteq T}} \sum_{i \in B^r(k)} \left[\prod_{j=0}^{h} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B \in \mathcal{B}^j:\, B \subseteq \mathcal{B}^{j+1}(i),\\ B \cap T \neq \emptyset}} v(B)} \right] \Delta_v(T)$$

$$= \frac{1}{v\big(B^r(k)\big)} \sum_{\substack{T \subseteq N, T \not\subseteq \mathcal{B}^h(k),\\ (\mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)) \subseteq T}} \left[\prod_{j=r}^{h} \frac{v\big(\mathcal{B}^j(k)\big)}{\sum_{\substack{B \in \mathcal{B}^j:\, B \subseteq \mathcal{B}^{j+1}(k),\\ B \cap T \neq \emptyset}} v(B)} \right.$$

$$\left. \cdot \sum_{i \in \mathcal{B}^r(k)} \prod_{j=0}^{r-1} \frac{v\big(\mathcal{B}^j(i)\big)}{\sum_{\substack{B \in \mathcal{B}^j:\, B \subseteq \mathcal{B}^{j+1}(i),\\ B \cap T \neq \emptyset}} v(B)} {}^{14} \right] \Delta_v(T)$$

$$= \frac{1}{v\big(B^r(k)\big)} \sum_{\substack{T \subseteq N, T \not\subseteq \mathcal{B}^h(k),\\ (\mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)) \subseteq T}} \left[\prod_{j=r}^{h} \frac{v\big(\mathcal{B}^j(k)\big)}{\sum_{\substack{B \in \mathcal{B}^j:\, B \subseteq \mathcal{B}^{j+1}(k),\\ B \cap T \neq \emptyset}} v(B)} \right] \Delta_v(T)$$

$$= \frac{1}{v\big(B^r(\ell)\big)} \sum_{\substack{T \subseteq N, T \not\subseteq \mathcal{B}^h(k),\\ T \subseteq (\mathcal{B}^r(k) \cup \mathcal{B}^r(\ell))}} \left[\prod_{j=r}^{h} \frac{v\big(\mathcal{B}^j(\ell)\big)}{\sum_{\substack{B \in \mathcal{B}^j:\, B \subseteq \mathcal{B}^{j+1}(\ell),\\ B \cap T \neq \emptyset}} v(B)} \right] \Delta_v(T)$$

$$= \sum_{i \in B^r(\ell)} \frac{Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) - Sh_i\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big)}{v\big(B^r(\ell)\big)}. \qquad \square$$

9.4 Proof of theorem 5.3

By theorem 5.2, we have only to show that φ is unique.

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$ arbitrary and φ an LS-value which satisfies all axioms of theorem 5.3. We proceed in several steps.

[14] If $r = 0$ we have an empty product, which is equal, by convention, to the multiplicative identity 1.

Step 1 We denote by $v_{\overline{\mathcal{B}}^+}$ the coalition function where $v_{\overline{\mathcal{B}}^+}(B) := v(B)$ for all $B \in \overline{\mathcal{B}}$, $v_{\overline{\mathcal{B}}^+}(N) := v(N)$ and all other coalitions $Q \subseteq N$ are not active in $v_{\overline{\mathcal{B}}^+}$.
By lemma 9.2 and lemma 9.3, all coalitions $\mathcal{T}^r \subseteq \mathcal{B}^r$ with related coalitions $T^r \subsetneq N$, $T^r \notin \overline{\mathcal{B}}$, are not active in the induced r-th level game $(\mathcal{B}^r, v^r_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}^r}) \in \mathcal{GL}_0^{\mathcal{B}^r}$, $0 \leq r \leq h$. Thus, by lemma 9.4 and possibly convention 9.1, all $B_k, B_\ell \in \mathcal{B}^r$ with $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ are weakly dependent in $v^r_{\overline{\mathcal{B}}^+}$, $0 \leq r \leq h$, and **PBC** can be applied. We use an induction I_1 on the size m, $0 \leq m \leq h+1$, $m := h+1-r$ and an arbitrary $i \in N$ to show that $\sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$ is unique for all r, $0 \leq r \leq h+1$.
Initialisation I_1: Let $m = 0$ and so $r = h+1$. By **E**, $\sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$ is unique.
Induction step I_1: Assume that $\sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$ is unique for an arbitary m', $0 \leq m' \leq h$, and so for an $r' := h+1-m'$, (IH_1). It follows, with $r := r'-1$,

$$
\begin{aligned}
\sum_{\substack{B \in \mathcal{B}^r,\\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}}) \underset{(\mathbf{PBC})}{=} & \sum_{\substack{B \in \mathcal{B}^r,\\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{v(B)}{v\big(\mathcal{B}^r(i)\big)} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}}) \underset{(IH_1)}{=} c_1 \\
\Leftrightarrow \quad \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}}) = & \; c_2,\ c_1, c_2 \in \mathbb{R} \text{ uniquely determined.}
\end{aligned}
$$

We obtain that φ is unique on all coalition functions $v_{\overline{\mathcal{B}}^+}$.

Step 2 We denote by $v_{\overline{\mathcal{B}}}$ the coalition function where $v_{\overline{\mathcal{B}}}(B) := v(B)$ for all $B \in \overline{\mathcal{B}}$ and all other coalitions $Q \subseteq N$ are not active in $v_{\overline{\mathcal{B}}}$. By lemma 3.2, all players $i \in N$ are loyal players and, by **L**, we have $\varphi_i(N, v_{\overline{\mathcal{B}}}, \underline{\mathcal{B}}) = v(\{i\})$, if $h = 0$, and $\varphi_i(N, v_{\overline{\mathcal{B}}}, \underline{\mathcal{B}}) = \varphi_i(\mathcal{B}^h(i), v_{\overline{\mathcal{B}}}, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)})$, otherwise. In the $(h-1)$-th cut of the restriction to $\mathcal{B}^h(i)$ we can apply step 1 and get that φ is unique on $v_{\overline{\mathcal{B}}}$.

Step 3 Let $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subsetneq B^{r+1}$, $S_r \nsubseteq B^r \in \mathcal{B}^r$, $0 \leq r \leq h$, and $\underline{\mathcal{B}_r}|_{B^{r+1}}$ the r-th cut level structure restricted to B^{r+1}. We denote by v_{S_r}, $v_{S_r} \in \mathcal{GL}_0^{B^{r+1}}$, the coalition function where $v_{S_r}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $\Delta_{v_{S_r}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq B^{r+1}$ are not active in v_{S_r}.
There exists, by lemma 9.2, exactly one coalition $\mathcal{T}^q \subseteq \mathcal{B}^q|_{B^{r+1}}$ with related coalition $T^q \subseteq B^{r+1}$ in the induced q-th level game $(\mathcal{B}^q|_{B^{r+1}}, v^q_{S_r}, \underline{\mathcal{B}^q_r}|_{B^{r+1}}) \in \mathcal{GL}_0^{\mathcal{B}^q|_{B^{r+1}}}$, $0 \leq q \leq r$, such that $S_r = S_{T^q}$ from lemma 9.2. We obtain, by lemma 9.3, $\Delta_{v^q_{S_r}}(\mathcal{T}^q) = \Delta_v(S_r)$. All coalitions $\mathcal{R}^q \subseteq \mathcal{B}^q|_{B^{r+1}}$ with related coalitions $R^q \subseteq B^{r+1}$, $R^q \notin \overline{\mathcal{B}_r}|_{B^{r+1}}$, $R^q \neq T^q$, are not active in $v^q_{S_r}$. Thus, by lemma 9.4 and convention 9.1, all $B_k, B_\ell \in \mathcal{B}^q$, $B_\ell \subseteq \mathcal{B}^{q+1}(B_k)$, and $B_k, B_\ell \subseteq T^q$ are weakly dependent in $v^q_{S_r}$ and **PBC** can be applied. By lemma 3.2, all players $i \in B^{r+1} \backslash S_r$ are loyal players in v_{S_r}. By **L** and step 2, we have

$$
\varphi_i(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) \text{ is unique for all } i \in B^{r+1} \backslash S_r. \tag{13}
$$

We use an induction I_2 on the size m, $0 \leq m \leq r$, for all levels q, $0 \leq q \leq r$, with $m := r - q$, $c_3, c_4, c_5, c_6 \in \mathbb{R}$ and an arbitrary $B^q \in \mathcal{B}^q|_{B^{r+1}}$ to show that $\sum_{j \in B^q} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}})$ is uniquely determined.
Initialisation I_2: Let $i \in S_r$ arbitrary, $m = 0$ and so $q = r$. We get

$$
\sum_{\substack{B \in \mathcal{B}^r,\\ B \subseteq T^r}} \sum_{j \in B} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) \underset{(\mathbf{PBC})}{=} \sum_{\substack{B \in \mathcal{B}^r,\\ B \subseteq T^r}} \frac{v(B)}{v\big(\mathcal{B}^r(i)\big)} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) \underset{\substack{(\mathbf{E})\\(13)}}{=} c_3
$$

$$\Leftrightarrow \qquad \sum_{j\in\mathcal{B}^r(i)} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) = c_4$$

and $\sum_{j\in B^r} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}})$ is unique for all $B^r \in \mathcal{B}^r|_{B^{r+1}}$.

Induction step I_2: Assume that $\sum_{j\in\mathcal{B}^{q'}} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}})$ is uniquely determined for all $B^{q'} \in \mathcal{B}^{q'}$ and $m' := r - q'$, $0 \le m' \le r-1$, (IH_2). It follows for an arbitrary $i \in S_r$, $m := m' + 1$ and so $q = q' - 1$,

$$\sum_{\substack{B\in\mathcal{B}^q,\\ B\subseteq T^q,\\ B\subseteq\mathcal{B}^{q+1}(i)}} \sum_{j\in B} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) \underset{(\mathbf{PBC})}{=} \sum_{\substack{B\in\mathcal{B}^q,\\ B\subseteq T^q,\\ B\subseteq\mathcal{B}^{q+1}(i)}} \frac{v(B)}{v\big(\mathcal{B}^q(i)\big)} \sum_{j\in\mathcal{B}^q(i)} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) \underset{\substack{(IH_2)\\(13)}}{=} c_5$$

$$\Leftrightarrow \qquad \sum_{j\in\mathcal{B}^q(i)} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}) = c_6$$

and $\sum_{j\in B^q} \varphi_j(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}})$ is unique for all $B^q \in \mathcal{B}^q|_{B^{r+1}}$. So φ is unique on v_{S_r}, too.

Step 4 Let $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subsetneq B^{r+1}$, $S_r \nsubseteq B^r \in \mathcal{B}^r$, $0 \le r \le h$, and $\underline{\mathcal{B}_r}|_{B^{r+1}}$ the r-th cut level structure restricted to B^{r+1}. We denote by $v_{S_r^+}$, $v_{S_r^+} \in \mathcal{GL}_0^{B^{r+1}}$, the coalition function where $v_{S_r^+}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $v_{S_r^+}(B^{r+1}) := v(B^{r+1})$, $\Delta_{v_{S_r^+}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq B^{r+1}$ are not active in $v_{S_r^+}$. Let $v' \in \mathcal{GL}_0^{B^{r+1}}$ the coalition function where $v'(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $\Delta_{v'}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq B^{r+1}$ are not active in v' and let $v'' \in \mathcal{GL}_0^{B^{r+1}}$ the coalition function where $v''(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $v''(B^{r+1}) := v(B^{r+1}) - v'(B^{r+1})$ and all other coalitions $R \subseteq B^{r+1}$ are not active in v''.

We have, by remark 9.5, $v_{S_r^+} = v' + v''$. We obtain, by step 3 for v', step 1 for v'' and **WA**, because $v'(B) = v''(B)$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, that φ is unique on $v_{S_r^+}$.

Step 5 Let $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subsetneq B^{r+1}$, $S_r \nsubseteq B^r \in \mathcal{B}^r$, $0 \le r \le \ell \le h$, and $\underline{\mathcal{B}_\ell}|_{\mathcal{B}^{\ell+1}(S_r)}$ the ℓ-th cut level structure restricted to $\mathcal{B}^{\ell+1}(S_r)$. We denote by $v_{S_r^\ell}$, $v_{S_r^\ell} \in \mathcal{GL}_0^{\mathcal{B}^{\ell+1}(S_r)}$, the coalition function where $v_{S_r^\ell}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_\ell}|_{\mathcal{B}^{\ell+1}(S_r)}$, $\Delta_{v_{S_r^\ell}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq \mathcal{B}^{\ell+1}(S_r)$ are not active in $v_{S_r^\ell}$ and we denote by $v_{S_r^{\ell+}}$, $v_{S_r^{\ell+}} \in \mathcal{GL}_0^{\mathcal{B}^{\ell+1}(S_r)}$, the coalition function where $v_{S_r^{\ell+}}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_\ell}|_{\mathcal{B}^{\ell+1}(S_r)}$, $v_{S_r^{\ell+}}\big(\mathcal{B}^{\ell+1}(S_r)\big) := v\big(\mathcal{B}^{\ell+1}(S_r)\big)$, $\Delta_{v_{S_r^{\ell+}}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq \mathcal{B}^{\ell+1}(S_r)$ are not active in $v_{S_r^{\ell+}}$. We use an induction I_3 on the size ℓ, $r \le \ell \le h$, to show that φ is unique on $v_{S_r^{\ell+}}$.

Initialisation I_3: Let $\ell = r$. Step 4 shows that φ is unique on $v_{S_r^{r+}}$ because $v_{S_r^{r+}} = v_{S_r^+}$ from step 4 and thus also on $v_{S_r^{\ell+}}$ if $h = 0$.

Induction step I_3: Assume that φ is unique on $v_{S_r^{\ell'+}}$, $r \le \ell' \le h-1$, $h \ge 1$ (IH_3). Let $\ell := \ell' + 1$. In the game $v_{S_r^\ell}$ all players are loyal by lemma 3.2. Thus, by **L** and (IH_3), φ is unique on $v_{S_r^\ell}$ and, because v is arbitrary, φ is also unique on the coalition function $v''' \in \mathcal{GL}_0^{\mathcal{B}^{\ell+1}(S_r)}$ where $v'''(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_\ell}|_{\mathcal{B}^{\ell+1}(S_r)}$, $\Delta_{v'''}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq \mathcal{B}^{\ell+1}(S_r)$ are not active in v'''. By step 1, φ is unique on the coalition function $v'''' \in \mathcal{GL}_0^{\mathcal{B}^{\ell+1}(S_r)}$, too, where $v''''(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_\ell}|_{\mathcal{B}^{\ell+1}(S_r)}$,

$v''''\big(\mathcal{B}^{\ell+1}(S_r)\big) := v\big(\mathcal{B}^{\ell+1}(S_r)\big) - v'''\big(\mathcal{B}^{\ell+1}(S_r)\big)$ and all other coalitions $R \subseteq \mathcal{B}^{\ell+1}(S_r)$ are not active in v''''.
We have, by remark 9.5, $v_{S_r^{\ell+}} = v''' + v''''$. So, by **WA**, φ is unique on $v_{S_r^{\ell+}}$, $0 \leq r \leq \ell \leq h$.

Step 6 For each coalition $S \in \Omega^N$, if $S \notin \overline{\mathcal{B}}$, we define $v_S(B) := \dfrac{v(B)}{2^{n-1}}$, $\Delta_{v_S}(S) := \Delta_v(S)$ and all other coalitions $Q \subseteq N$ are not active in v_S and, if $S \in \overline{\mathcal{B}}$, we define $v_S(B) := \dfrac{v(B)}{2^{n-1}}$ and all other coalitions $Q \subseteq N$ are not active in v_S. By remark 9.5, we have $v = \sum_{S\in\Omega^N} v_S$. By step 1, 2 and 5 φ is unique on v_S for all $S \in \Omega^N$. Thus, by **WA**, φ is uniquely on v. $\square$

9.5 Proof of proposition 5.9

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B \in \mathcal{B}^r$, $0 \leq r \leq h$, and $(N^B, v^B, \underline{\mathcal{B}^B}) \in \mathcal{GL}^{N^B}$ a CS-game to $(N, v, \underline{\mathcal{B}})$ with related components $B', B'' \in \underline{\mathcal{B}^B}$.
If $r = 0$, $\{k\} := B'$ and $\{\ell\} := B''$, $k, \ell \in N^B$, we get

$$
\begin{aligned}
Sh_i^{pSL}(N, v, \underline{\mathcal{B}}) &\underset{\substack{Def.\\5.1}}{=} \sum_{T\subseteq N,\, T\ni i} K_{v,T}(i)\Delta_v(T)\\
&= \sum_{S\subseteq N\setminus B,\, S\ni i} K_{v,S}\Delta_v(S) + \sum_{S\subseteq N\setminus B,\, S\ni i} K_{v,S\cup B}\Delta_v(S\cup B)\\
&\underset{\substack{Rem.5.7\\Lem.9.4}}{=} \sum_{\substack{S\subseteq N^B\setminus\{k,\ell\},\\ S\ni i}} K_{v^B,S}\Delta_{v^B}(S) + \sum_{\substack{S\subseteq N^B\setminus\{k,\ell\},\\ S\ni i}} K_{v^B,S\cup\{k,\ell\}}\Delta_{v^B}(S\cup\{k,\ell\})\\
&= \sum_{T\subseteq N^B,\, T\ni i} K_{v^B,T}\Delta_{v^B}(T) \underset{\substack{Def.\\5.1}}{=} Sh_i^{pSL}(N^B, v^B, \underline{\mathcal{B}^B}) \text{ for all } i \in N\setminus B.
\end{aligned}
$$

Thus we have, by remark 5.7 and lemma 9.4, also in the r-th level game, $0 \leq r \leq h$,

$$
Sh^{pSL}_{\mathcal{B}^r(i)}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) = Sh^{pSL}_{(\mathcal{B}^B)^r(i)}\big((\mathcal{B}^B)^r, (v^B)^r, \underline{(\mathcal{B}^B)^r}\big) \text{ for all } i \in N\setminus B
$$

and the claim follows by **LG**. $\square$

9.6 Proof of lemma 5.10

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$, φ an LS-value which satisfies **E** and **CS**, $B_1, B_2 \in \mathcal{B}^r$, $0 \leq r \leq h$, $B_2 \subseteq \mathcal{B}^{r+1}(B_1)$, two symmetric components in the r-th level game v^r and, w.l.o.g., $\mathcal{B}^r = \{B_1, B_2, ..., B_t\}$. We substitute component B_1 according to **CS** by two new components, component B_{t+1} and component B_{t+2}, $(\mathcal{B}^{B_1})^r := \{B_2, B_3, ..., B_t, B_{t+1}, B_{t+2}\}$ and obtain

$$
\sum_{i\in B_2} \varphi_i(N^{B_1}, v^{B_1}, \underline{\mathcal{B}^{B_1}}) \underset{(\mathbf{CS})}{=} \sum_{i\in B_2} \varphi_i(N, v, \underline{\mathcal{B}}), \tag{14}
$$

and, if we substitute component B_2 according to **CS** by the same components as before, B_{t+1} and B_{t+2}, instead, $(\mathcal{B}^{B_2})^r := \{B_1, B_3, ..., B_t, B_{t+1}, B_{t+2}\}$, we have

$$
\sum_{i\in B_1} \varphi_i(N^{B_2}, v^{B_2}, \underline{\mathcal{B}^{B_2}}) \underset{(\mathbf{CS})}{=} \sum_{i\in B_1} \varphi_i(N, v, \underline{\mathcal{B}}), \tag{15}
$$

where we choose $v^{B_2}(B_{t+1}) := v^{B_1}(B_{t+1})$ and $v^{B_2}(B_{t+2}) := v^{B_1}(B_{t+2})$.

In the same manner we substitute now in the game $(N^{B_1}, v^{B_1}, \underline{\mathcal{B}^{B_1}})$ component B_2 by component B_{t+3} and component B_{t+4}, and analogous in the game $(N^{B_2}, v^{B_2}, \underline{\mathcal{B}^{B_2}})$ component B_1 by the same components as before, B_{t+3} and B_{t+4}, and choose $v^{B_2{}^{B_1}}(B_{t+3}) := v^{B_1{}^{B_2}}(B_{t+3})$ and $v^{B_2{}^{B_1}}(B_{t+4}) := v^{B_1{}^{B_2}}(B_{t+4})$. We have, by $B_2 \subseteq \mathcal{B}^{r+1}(B_1)$, in the q-th level game $(\mathcal{B}^{B_1{}^{B_2}})^q = (\mathcal{B}^{B_2{}^{B_1}})^q$, $r \leq q \leq h+1$. So, by remark 5.6, we can choose the structure of the components of all levels and the worths of the coalitions, which are arbitrary in the used CS-games[15], such that we have $\big(N^{B_1{}^{B_2}}, v^{B_1{}^{B_2}}, \underline{\mathcal{B}^{B_1{}^{B_2}}}\big) = \big(N^{B_2{}^{B_1}}, v^{B_2{}^{B_1}}, \underline{\mathcal{B}^{B_2{}^{B_1}}}\big)$ and get, by **E** and remark 5.8,

$$\begin{aligned}
&\sum_{i\in B_{t+3}} \varphi_i\big(N^{B_1{}^{B_2}}, v^{B_1{}^{B_2}}, \underline{\mathcal{B}^{B_1{}^{B_2}}}\big) + \sum_{i\in B_{t+4}} \varphi_i\big(N^{B_1{}^{B_2}}, v^{B_1{}^{B_2}}, \underline{\mathcal{B}^{B_1{}^{B_2}}}\big)\\
&\qquad\qquad = \sum_{i\in B_2} \varphi_i(N^{B_1}, v^{B_1}, \underline{\mathcal{B}^{B_1}}) \underset{(14)}{=} \sum_{i\in B_2} \varphi_i(N, v, \underline{\mathcal{B}}),\\
&\sum_{i\in B_{t+3}} \varphi_i\big(N^{B_2{}^{B_1}}, v^{B_2{}^{B_1}}, \underline{\mathcal{B}^{B_2{}^{B_1}}}\big) + \sum_{i\in B_{t+4}} \varphi_i\big(N^{B_2{}^{B_1}}, v^{B_2{}^{B_1}}, \underline{\mathcal{B}^{B_2{}^{B_1}}}\big)\\
&\qquad\qquad = \sum_{i\in B_1} \varphi_i(N^{B_2}, v^{B_2}, \underline{\mathcal{B}^{B_2}}) \underset{(15)}{=} \sum_{i\in B_1} \varphi_i(N, v, \underline{\mathcal{B}})
\end{aligned}$$

and hence $\sum_{i\in B_1} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i\in B_2} \varphi_i(N, v, \underline{\mathcal{B}})$ and **SymBC** is shown. □

9.7 Proof of lemma 5.11

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N_{0_\mathbb{Q}}$, φ an LS-value which satisfies **E** and **CS** and therefore, by lemma 5.10, also **SymBC**, $B_1, B_2 \in \mathcal{B}^r$, $0 \leq r \leq h$, $B_2 \subseteq \mathcal{B}^{r+1}(B_1)$, two weakly dependent components in the r-th level game v^r.

Due to $v(B_1), v(B_2) \in \mathbb{Q}_{++}$ the worths of the components $v(B_k)$, $k \in \{1,2\}$, can be written as a fraction

$$v(B_k) = \frac{p_k}{q_k} \quad \text{with} \quad p_k, q_k \in \mathbb{N}.$$

We choose a main denominator q of these two fractions by $q := q_1 q_2$. With $z_1 := p_1 q_2$ and $z_2 := p_2 q_1$ we get

$$v(B_1) = \frac{z_1}{q} \quad \text{and} \quad v(B_2) = \frac{z_2}{q}. \tag{16}$$

Applying **CS** (repeatedly) to $(N, v, \underline{\mathcal{B}})$ and the two components B_1, B_2 we can get the LS-game $(N', v', \underline{\mathcal{B}'})$ where each component B_k, $k \in \{1,2\}$, is "substituted" by z_k components B_{k_1} to $B_{k_{(z_k)}}$, such that $(\mathcal{B}')^r = (\mathcal{B}^r \backslash \{B_1, B_2\}) \cup \{B_{1_m} : 1 \leq m \leq z_1\} \cup \{B_{2_m} : 1 \leq m \leq z_2\}$ and each component $B_\ell \in (\mathcal{B}')^r \backslash (\mathcal{B}^r \backslash \{B_1, B_2\})$ gets a worth $v'(B_\ell) = \frac{1}{q}$ where $\big|(\mathcal{B}')^r \backslash (\mathcal{B}^r \backslash \{B_1, B_2\})\big| = z_1 + z_2$ and $v(B_k) = \sum_{1\leq m\leq z_k} v'(B_{k_m})$, $k \in \{1,2\}$.

[15] E. g., regard CS-games where the substituting components are singletons, all levels lower then r contain only singletons as elements and all players are dependent outside the r-th component.

All components $B_\ell \in (\mathcal{B}')^r \backslash (\mathcal{B}^r \backslash \{B_1, B_2\})$ are symmetric in the r-th level game $(v')^r$ and subsets of the same component in the $(r+1)$-th level. Hence follows, by **SymBC**,

$$\frac{1}{z_1} \sum_{1 \le m \le z_1} \sum_{i \in B_{1_m}} \varphi_i(N', v', \underline{\mathcal{B}'}) = \frac{1}{z_2} \sum_{1 \le m \le z_2} \sum_{i \in B_{2_m}} \varphi_i(N', v', \underline{\mathcal{B}'}).$$

We obtain, by remark 5.8 and (16),

$$\sum_{i \in B_1} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{v(B_1)} = \sum_{i \in B_2} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{v(B_2)}. \qquad \square$$

9.8 Proof of theorem 6.2

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}_{0+}^N$ and $A_{v,T}$, $A_{v,T}^r$ the expressions according to def. 5.1.

• **E, D, L, H, PWC, PTCC**: The partial proofs are analogous to the partial proofs of the proof of theorem 5.2 and therefore omitted.

• **CWA**: Let $v'(S) = c \cdot v(S)$ for all $S \subseteq B$, $B \in \mathcal{B}^h$, $c > 0$. It follows $A_{v,T}^r = A_{v',T}^r$ and thus $A_{v,T} = A_{v',T}$ and we can transfer the proof from **WA** from theorem 5.2.

• **DO**: It is well-known that each coalition $T \in \Omega^N$ containing a dummy player $j \in N$ in v is not active in v if $T \neq \{j\}$. In eq. (7) we have only to consider active coalitions. For these coalitions is no change in the coefficients $A_{v,T}$ in the restriction. Thus we get $Sh_i^{pAL}(N, v, \underline{\mathcal{B}}) = Sh_i^{pAL}(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}})$ for all $i \in N \backslash \{j\}$.

• **LO**: Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $j \in N$ a loyal player in v and $i \in N \backslash \{j\}$.

In eq. (7) we have only to consider active coalitions. For these coalitions is no change in the coefficients $A_{v,T}$ in the following restrictions. We distinguish two cases.

Case 1 $\mathcal{B}^h(i) \neq \mathcal{B}^h(j)$. We have

$$\begin{aligned}
Sh_i^{pAL}(N, v, \underline{\mathcal{B}}) &\underset{\substack{Def.\\6.1}}{=} \sum_{T \subseteq N, T \ni i} A_{v,T}(i) \Delta_v(T) \\
&= \sum_{\substack{T \subseteq \mathcal{B}^h(j),\\ T \ni i}} A_{v,T}(i) \Delta_v(T) + \sum_{\substack{T \subseteq N, T \ni i\\ T \not\subseteq \mathcal{B}^h(j), T \ni j}} A_{v,T}(i) \Delta_v(T) + \sum_{\substack{T \subseteq N, T \ni i\\ T \not\subseteq \mathcal{B}^h(j), j \notin T}} A_{v,T}(i) \Delta_v(T) \\
&\underset{\substack{Lem.\\3.2}}{=} \sum_{\substack{T \subseteq N, T \ni i\\ T \not\subseteq \mathcal{B}^h(j), j \notin T}} A_{v,T}(i) \Delta_v(T) = \sum_{\substack{T \subseteq N \backslash \{j\},\\ T \ni i}} A_{v,T}(i) \Delta_v(T) \\
&= Sh_i^{pAL}\big(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}}\big).
\end{aligned}$$

Case 2 $\mathcal{B}^h(i) = \mathcal{B}^h(j)$. Regard, in this case, we always have $h \geq 1$. It follows

$$\begin{aligned}
Sh_i^{pAL}(N, v, \underline{\mathcal{B}}) &\underset{\substack{Def.\\6.1}}{=} \sum_{\substack{T \subseteq \mathcal{B}^h(j),\\ T \ni i}} A_{v,T}(i)\Delta_v(T) + \sum_{\substack{T \subseteq N, T \ni i\\ T \not\subseteq \mathcal{B}^h(j), T \ni j}} A_{v,T}(i)\Delta_v(T) + \sum_{\substack{T \subseteq N, T \ni i\\ T \not\subseteq \mathcal{B}^h(j), j \notin T}} A_{v,T}(i)\Delta_v(T)\\
&\underset{\substack{Lem.\\3.2}}{=} \sum_{\substack{T \subseteq \mathcal{B}^h(i),\\ T \ni i}} A_{v,T}(i)\Delta_v(T) + \sum_{\substack{T \subseteq N, T \ni i\\ T \not\subseteq \mathcal{B}^h(i), j \notin T}} A_{v,T}(i)\Delta_v(T)\\
&= \sum_{\substack{T \subseteq \mathcal{B}^h(i),\\ T \ni i}} A_{v,T}(i)\Delta_v(T) + \sum_{\substack{T \subseteq N \backslash \{j\},\\ T \ni i}} A_{v,T}(i)\Delta_v(T) - \sum_{\substack{T \subseteq \mathcal{B}^h(i) \backslash \{j\},\\ T \ni i}} A_{v,T}(i)\Delta_v(T)\\
&= Sh_i^{pAL}\big(\mathcal{B}^h(i), v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)}\big) + Sh_i^{pAL}\big(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}}\big)\\
&\quad - Sh_i^{pAL}\big(\mathcal{B}^h(i) \backslash \{j\}, v, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i) \backslash \{j\}}\big)
\end{aligned}$$

and **LO** is shown. $\square$

9.9 Proof of theorem 6.3

By theorem 6.2 we have only to show that φ is unique.

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_{0+}^N$ and φ an LS-value which satisfies all axioms of theorem 6.3 and **L** by lemma 9.6.

If $|N| = 1$, φ is unique by **E**. Let now $|N| \geq 2$, $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subseteq B^{r+1}$, $S_r \not\subseteq B^r \in \mathcal{B}^r$, $0 \leq r \leq h$, and $\underline{\mathcal{B}_r}|_{B^{r+1}}$ the r-th cut level structure restricted to B^{r+1}. We denote by v_{S_r}, $v_{S_r} \in \mathcal{GL}_{0+}^{B^{r+1}}$, the coalition function where $v_{S_r}(T) := v(T)$ for all $T \subseteq B^r$ and all $B^r \in \mathcal{B}^r, B^r \subseteq B^{r+1}$, $v_{S_r}(S_r) = c_{S_r}$, $c_{S_r} \in \mathbb{R}$ arbitrary, and all other coalitions $R \subseteq B^{r+1}$ are not active in v_{S_r}.

We use an induction I_1 on the size r, $0 \leq r \leq h$, to show that φ is unique for all LS-games $\big(B^{r+1}, v, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$.

Initialisation I_1: Let $r = 0$. By lemma 3.2, all players $i \in B^{r+1} \backslash S_r$ are loyal players in v_{S_r}. So, by **L**, $\varphi_i\big(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$ is unique for all $i \in B^{r+1} \backslash S_r$. We have, by lemma 3.4, all $i \in S_r$ are tc-dependent in $\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)$. We get for an arbitrary $i \in S_r$

$$\sum_{j \in S_r} \varphi_j\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) \underset{\textbf{(PTCC)}}{=} \sum_{j \in S_r} \frac{v(\{j\})}{v(\{i\})} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) \underset{\textbf{(E)}}{=} c_{S_r}$$

and $\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)$ is unique for all $i \in S_r$. Thus, by **LO**, $\varphi_i\big(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$ is unique for all $i \in S_r$ and so for all games $\big(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$. Because v and c_{S_r} are arbitrary and by **CWA** and remark 9.5, we obtain that φ is unique on all games $\big(B^{r+1}, v, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$ if $r = 0$ and so on all games $(N, v, \underline{\mathcal{B}_0}) \in \mathcal{GL}_{0+}^N$.

In the following we require $h \geq 1$.

Induction step I_1: Assume that φ is unique on all games $\big(B^{r'+1}, v, \underline{\mathcal{B}_{r'}}|_{B^{r'+1}}\big), B^{r'+1} \in \mathcal{B}^{r'+1}$, $0 \leq r' \leq h-1$, (IH_1). Let $r := r'+1$. By lemma 3.2, all players $i \in B^{r+1} \backslash S_r$ are loyal players in v_{S_r}. So, by **L** and (IH_1), $\varphi_i\big(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$ is unique for all $i \in B^{r+1} \backslash S_r$ and we have, by lemma 3.4, all $i \in S_r$ are tc-dependent in $\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)$.

We use an induction I_2 on the size m, $0 \leq m \leq r$, to show that $\sum_{i \in B \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)$ is unique for all $B \in \mathcal{B}^q$, $B \cap S_r \neq \emptyset$ and $q := r - m$.

Initialisation I_2: Let $m = 0$ and so $q = r$, $B_k, B_\ell \in \mathcal{B}^r$, $B_k, B_\ell \cap S_r \neq \emptyset$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}$. We have, by **PTCC**, probably using convention 9.1,

$$\sum_{i \in B_k \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) - \varphi_i\big(B^k \cap S_r, v_{S_r}, \underline{\mathcal{B}_{r-1}}|_{B_k \cap S_r}\big)}{v(B_k \cap S_r)}$$
$$= \sum_{i \in B_\ell \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) - \varphi_i\big(B_\ell \cap S_r, v_{S_r}, \underline{\mathcal{B}_{r-1}}|_{B_\ell \cap S_r}\big)}{v(B_\ell \cap S_r)}$$
$$\underset{(IH_1)}{\Leftrightarrow} \sum_{i \in B_k \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)}{v(B_k \cap S_r)} = \sum_{i \in B_\ell \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)}{v(B_\ell \cap S_r)} + c_1.$$

We get

$$\sum_{\substack{B \in \mathcal{B}^r,\\ B \cap S_r \neq \emptyset}} \sum_{i \in B \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) = \sum_{\substack{B \in \mathcal{B}^r,\\ B \cap S_r \neq \emptyset}} \frac{v(B \cap S_r)}{v(B_k \cap S_r)} \sum_{i \in B_k \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) + c_2 \underset{\textbf{(E)}}{=} c_3$$
$$\Leftrightarrow \quad \sum_{i \in B_k \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) = c_4.$$

Induction step I_2: Assume that $\sum_{i \in B \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)$, $B \in \mathcal{B}^{q'}$, $B \cap S_r \neq \emptyset$ is unique for an arbitrary m', $0 \leq m' \leq r-1$, and so for a $q' := r - m'$, $1 \leq q' \leq r$, (IH_2). Let $m = m' + 1$ and so $q = r - m' - 1$, $B_k, B_\ell \in \mathcal{B}^q$, $B_\ell \subseteq \mathcal{B}^{q+1}(B_k)$, $B_k, B_\ell \cap S_r \neq \emptyset$ and $c_5, c_6, c_7, c_8 \in \mathbb{R}$. We have, by **PTCC**, probably using convention 9.1,

$$\sum_{i \in B_k \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) - \varphi_i\big(\mathcal{B}^r(B_k) \cap S_r, v_{S_r}, \underline{\mathcal{B}_{r-1}}|_{\mathcal{B}^r(B_k) \cap S_r}\big)}{v(B_k \cap S_r)}$$
$$= \sum_{i \in B_\ell \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) - \varphi_i\big(\mathcal{B}^r(B_\ell) \cap S_r, v_{S_r}, \underline{\mathcal{B}_{r-1}}|_{\mathcal{B}^r(B_\ell) \cap S_r}\big)}{v(B_\ell \cap S_r)}$$
$$\underset{(IH_1)}{\Leftrightarrow} \sum_{i \in B_k \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)}{v(B_k \cap S_r)} = \sum_{i \in B_\ell \cap S_r} \frac{\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)}{v(B_\ell \cap S_r)} + c_5.$$

We get

$$\sum_{\substack{B \in \mathcal{B}^q,\\ B \cap S_r \neq \emptyset,\\ B \subseteq \mathcal{B}^{q+1}(B_k)}} \sum_{i \in B \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) = \sum_{\substack{B \in \mathcal{B}^q,\\ B \cap S_r \neq \emptyset,\\ B \subseteq \mathcal{B}^{q+1}(B_k)}} \frac{v(B \cap S_r)}{v(B_k \cap S_r)} \sum_{i \in B_k \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) + c_6 \underset{(IH_2)}{=} c_7$$

$$\Leftrightarrow \quad \sum_{i \in B_k \cap S_r} \varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big) = c_8$$

and $\varphi_i\big(S_r, v_{S_r}, \underline{\mathcal{B}_r}|_{S_r}\big)$ is unique for all $i \in S_r$, $0 \leq r \leq h$. Thus, by **LO**, $\varphi_i\big(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$ is unique for all $i \in S_r$, $0 \leq r \leq h$, and so for all games $\big(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}}\big)$. Because v and c_{S_r} are arbitrary and by **CWA** and remark 9.5, we obtain that φ is unique on all games $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N_{0+}$. $\square$

9.10 Proof of theorem 7.3

By theorem 7.2 we have only to show that φ is unique.

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h} \in \mathcal{L}^N$, $v \in \mathcal{GL}_0^N$ and φ an LS-value which satisfies all axioms of theorem 7.3 and **L** by lemma 9.7. We proceed in several steps.

Step 1 We denote by $v_{\overline{\mathcal{B}}^+}$ the coalition function where $v_{\overline{\mathcal{B}}^+}(B) := v(B)$ for all $B \in \overline{\mathcal{B}}$, $v_{\overline{\mathcal{B}}^+}(N) := v(N)$ and all other coalitions $Q \subseteq N$ are not active in $v_{\overline{\mathcal{B}}^+}$. Let $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$. By lemma 3.4 and convention 9.1 all $i \in B_k \cup B_\ell$ are rc-dependent in $v_{\overline{\mathcal{B}}^+}$, $0 \leq r \leq h$, and **PWC** can be applied. We use an induction I_1 on the size $m := h + 1 - r$, $0 \leq m \leq h + 1$, and an arbitrary $i \in N$ to show that $\sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$ is unique for all r, $0 \leq r \leq h+1$.

Initialisation I_1: Let $m = 0$ and so $r = h + 1$. By **E**, $\sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$ is unique.

Induction step I_1: Assume that $\sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$ is unique for an arbitary $m' := h + 1 - r'$, $0 \leq m' \leq h$, (IH_1). Let $c_1, c_2 \in \mathbb{R}$, $m = m' + 1$ and so $r = r' - 1$. It follows

$$\sum_{\substack{B \in \mathcal{B}^r, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}}) \underset{(\mathbf{PWC})}{=} \sum_{\substack{B \in \mathcal{B}^r, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{v(B)}{v\big(\mathcal{B}^r(i)\big)} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}}) \underset{(IH_1)}{=} c_1$$

$$\Leftrightarrow \quad \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}}) = c_2.$$

We obtain that φ is uniquely defined on the game $(N, v_{\overline{\mathcal{B}}^+}, \underline{\mathcal{B}})$.

Step 2 We denote by $v_{\overline{\mathcal{B}}}$ the coalition function where $v_{\overline{\mathcal{B}}}(B) := v(B)$ for all $B \in \overline{\mathcal{B}}$ and all other coalitions $Q \subseteq N$ are not active in $v_{\overline{\mathcal{B}}}$. By lemma 3.2, all players $i \in N$ are loyal players and, by **L**, we have $\varphi_i(N, v_{\overline{\mathcal{B}}}, \underline{\mathcal{B}}) = v(\{i\})$, if $h = 0$, and $\varphi_i(N, v_{\overline{\mathcal{B}}}, \underline{\mathcal{B}}) = \varphi_i(\mathcal{B}^h(i), v_{\overline{\mathcal{B}}}, \underline{\mathcal{B}_{h-1}}|_{\mathcal{B}^h(i)})$, otherwise. In the $(h-1)$-th cut of the restriction to $\mathcal{B}^h(i)$ we can apply step 1 and get that φ is unique on $v_{\overline{\mathcal{B}}}$.

Step 3 Let $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subsetneq B^{r+1}$, $S_r \nsubseteq B^r \in \mathcal{B}^r$, $0 \leq r \leq h$, and $\underline{\mathcal{B}_r}|_{B^{r+1}}$ the r-th cut level structure restricted to B^{r+1}. We denote by v_{S_r}, $v_{S_r} \in \mathcal{GL}_0^{B^{r+1}}$, the coalition function where $v_{S_r}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $\Delta_{v_{S_r}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq B^{r+1}$ are not active in v_{S_r}.

By lemma 3.2, all players $i \in B^{r+1} \backslash S_r$ are loyal players in v_{S_r}. By **L** and step 2 we have that $\varphi_i(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}})$ is unique for all $i \in B^{r+1} \backslash S_r$ and, by **ILO**, for all $i \in S_r$, because we can apply step 1 on the internally induced restrictions. So φ is uniquely defined on $(B^{r+1}, v_{S_r}, \underline{\mathcal{B}_r}|_{B^{r+1}})$.

Step 4 Let $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subsetneq B^{r+1}$, $S_r \nsubseteq B^r \in \mathcal{B}^r$, $0 \leq r \leq h$, and $\underline{\mathcal{B}_r}|_{B^{r+1}}$ the r-th cut level structure restricted to B^{r+1}. We denote by $v_{S_r^+}$, $v_{S_r^+} \in \mathcal{GL}_0^{B^{r+1}}$, the coalition function where $v_{S_r^+}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $v_{S_r^+}(B^{r+1}) := v(B^{r+1})$, $\Delta_{v_{S_r^+}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq B^{r+1}$ are not active in $v_{S_r^+}$.

Let $v' \in \mathcal{GL}_0^{B^{r+1}}$ the coalition function where $v'(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $\Delta_{v'}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq B^{r+1}$ are not active in v' and let $v'' \in \mathcal{GL}_0^{B^{r+1}}$ the coalition function where $v''(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, $v''(B^{r+1}) := v(B^{r+1}) - v'(B^{r+1})$ and all other coalitions $R \subseteq B^{r+1}$ are not active in v''.

We have, by remark 9.5, $v_{S_r^+} = v' + v''$. We obtain, by step 3 for v', step 1 for v'' and **WA**, because $v'(B) = v''(B)$ for all $B \in \overline{\mathcal{B}_r}|_{B^{r+1}}$, that φ is unique on $(B^{r+1}, v_{S_r^+}, \underline{\mathcal{B}_r}|_{B^{r+1}})$.

Step 5 Let $B^{r+1} \in \mathcal{B}^{r+1}$, $S_r \subsetneq B^{r+1}$, $S_r \nsubseteq B^r \in \mathcal{B}^r$, $0 \le r \le q \le h$, and $\underline{\mathcal{B}_q}|_{\mathcal{B}^{q+1}(S_r)}$ the q-th cut level structure restricted to $\mathcal{B}^{q+1}(S_r)$. By $v_{\underline{S_r^q}}, v_{S_r^q} \in \mathcal{GL}_0^{\mathcal{B}^{q+1}(S_r)}$, we denote the coalition function where $v_{S_r^q}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_q}|_{\mathcal{B}^{q+1}(S_r)}$, $\Delta_{v_{S_r^q}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq \mathcal{B}^{q+1}(S_r)$ are not active in $v_{S_r^q}$ and by $v_{S_r^{q+}}, v_{S_r^{q+}} \in \mathcal{GL}_0^{\mathcal{B}^{q+1}(S_r)}$, the coalition function where $v_{S_r^{q+}}(B) := v(B)$ for all $B \in \overline{\mathcal{B}_q}|_{\mathcal{B}^{q+1}(S_r)}$, $v_{S_r^{q+}}\big(\mathcal{B}^{q+1}(S_r)\big) := v\big(\mathcal{B}^{q+1}(S_r)\big)$, $\Delta_{v_{S_r^{q+}}}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq \mathcal{B}^{q+1}(S_r)$ are not active in $v_{S_r^{q+}}$. We use an induction I_2 on the size q, $r \le q \le h$, to show that φ is unique on $v_{S_r^{q+}}$.

Initialisation I_2: Let $q = r$. Step 4 shows that φ is unique on $v_{S_r^{r+}}$ because $v_{S_r^{r+}} = v_{S_r^+}$.

Induction step I_2: Assume that φ is unique on $v_{S_r^{\tilde{q}+}}$, $r \le \tilde{q} \le h-1$, (IH_2). Let $q := \tilde{q}+1$. In the game $v_{S_r^q}$ all players are loyal by lemma 3.2. Thus, by **L** and (IH_2), φ is unique on $v_{S_r^q}$ and, because v is arbitrary, φ is also unique on the coalition function $v' \in \mathcal{GL}_0^{\mathcal{B}^{q+1}(S_r)}$ where $v'(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_q}|_{\mathcal{B}^{q+1}(S_r)}$, $\Delta_{v'}(S_r) := \Delta_v(S_r)$ and all other coalitions $R \subseteq \mathcal{B}^{q+1}(S_r)$ are not active in v'. By step 1, φ is unique on the coalition function $v'' \in \mathcal{GL}_0^{\mathcal{B}^{q+1}(S_r)}$, too, where $v''(B) := \dfrac{v(B)}{2}$ for all $B \in \overline{\mathcal{B}_q}|_{\mathcal{B}^{q+1}(S_r)}$, $v''\big(\mathcal{B}^{q+1}(S_r)\big) := v\big(\mathcal{B}^{q+1}(S_r)\big) - v'\big(\mathcal{B}^{q+1}(S_r)\big)$ and all other coalitions $R \subseteq \mathcal{B}^{q+1}(S_r)$ are not active in v''.

We have, by remark 9.5, $v_{S_r^{q+}} = v' + v''$. So, by **WA**, φ is unique on $v_{S_r^{q+}}$ for $0 \le r \le q \le h$.

Step 6 For each coalition $S \in \Omega^N$, if $S \notin \overline{\mathcal{B}}$, we define $v_S(B) := \dfrac{v(B)}{2^{n-1}}$, $\Delta_{v_S}(S) := \Delta_v(S)$ and all other coalitions $Q \subseteq N$ are not active in v_S and, if $S \in \overline{\mathcal{B}}$, we define $v_S(B) := \dfrac{v(B)}{2^{n-1}}$ and all other coalitions $Q \subseteq N$ are not active in v_S. By remark 9.5, we have $v = \sum_{S \in \Omega^N} v_S$. By step 1, 2 and 5 φ is unique on $(N, v_S, \underline{\mathcal{B}})$ for all $S \in \Omega^N$. Thus, by **WA**, φ is uniquely defined on all $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}_0^N$. □

9.11 Logical independence

All axiomatizations must hold also for a trivial level structure $\underline{\mathcal{B}_0}$ and in this case all axioms used for axiomatization coincide with axioms for TU-values. So the given axiomatizations coincide in this case with axiomatizations of the proportional Shapley value. These axiomatizations of the proportional Shapley value use logical independent axioms what is is well-known or easy to prove. Therefore all axioms for LS-values are also logical independent in the given axiomatizations.

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